Quiz #2 solution Fall 2003-04

Problem #1

The bit rate in PCM is:

$$BR_{PCM} = \text{sampling rate} \times \log_2 L$$

$$= 2 \times 4000 \times \log_2 (256 = 2^8) = 64{,}000 \text{ bits/sec.}$$

The sampling rate in DM needs, in principle, to be as high as possible in order to justify the use of two quantization levels. Hence, since the replacement of PCM by DM should not lead to a higher transmission bandwidth, than the sampling rate that is needed in DM is 64,000 samples/sec = 8 × Nyquist rate. This is equal to the bit rate in DM and hence resulting in the same transmission bandwidth as in PCM.

Problem #2

The condition that needs to be satisfied to avoide slope overload distortion is:

$$\frac{\Delta}{T_s} \geq \max \left| \frac{dm(t)}{dt} \right| = \text{maximum slope of the message signal.}$$

$$\Rightarrow \frac{1}{T_s} \geq \frac{\text{maximum slope of } m(t)}{\Delta} = \frac{100}{2} = 50 \text{ samples/se}$$

But the lowest sampling rate that can be used is equal to the Nyquist rate, i.e., 8,000 samples/sec. Hence going from the sampling rate of 64,000 samples/sec to 8,000 samples/sec, which can be used in principle but not in practice, then the maximum saving in the transmission bandwidth is 64,000 − 8,000 = 56,000 Hz or 56 KHz.

Problem # 3

For a maximum saving in the transmission bandwidth equal to 50 KHz to be achieved, we need:

$$\frac{1}{T_s} = 64,000 - 50,000 = 14,000 \text{ samples/sec.}$$

$$= \frac{1}{\Delta} \times \text{maximum slope of } m(t) = \frac{100}{\Delta}$$

$$\Rightarrow \Delta = \frac{100}{14,000} = 7.14 \text{ mV.}$$

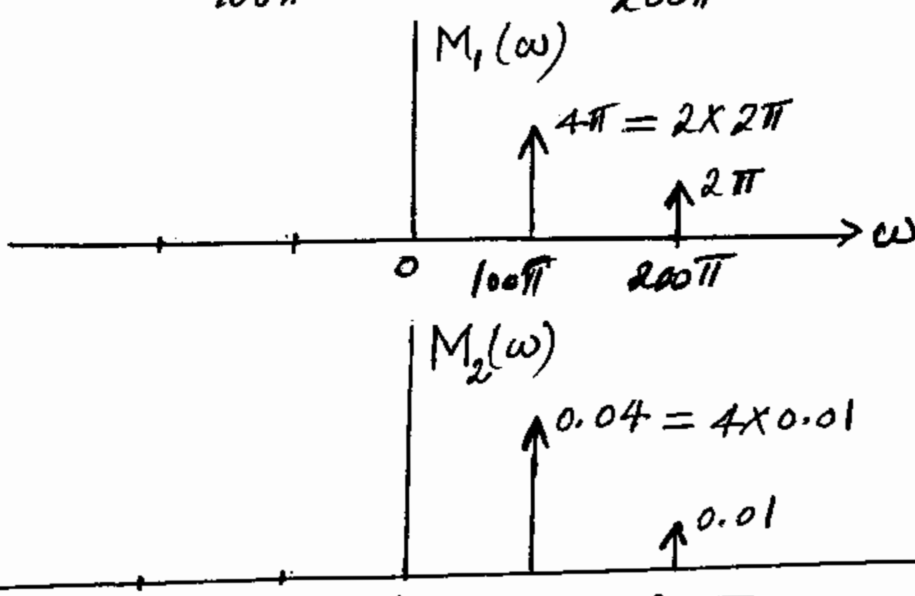
The minimum achievable transmission bandwidth is 14 KHz which corresponds to 14 K samples/sec. Hence, the sampling rate is equal to 1.75 × Nyquist rate of sampling.

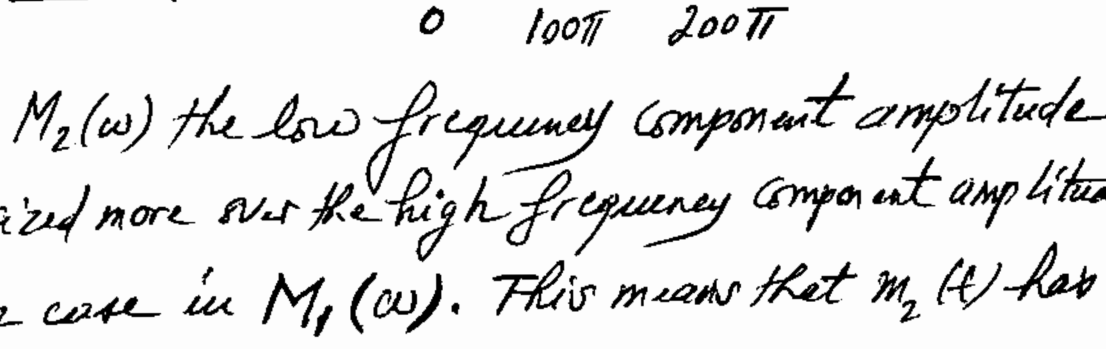
The sampling rate of 1.75 x Nyquist rate is not sufficient for use in DM to justify the fact that we only have two quantization levels. Actually, in DM the sampling rate needs to be much higher than the Nyquist rate so as to obtain a very high correlation between adjacent samples and hence provide a very small dynamic range of the error signal available at the input of the DM quantizer.

Problem #4

$$m_1(t) = 4\cos(100\pi t) + 2\cos(200\pi t)$$

$$m_2(t) = \frac{4}{100\pi}\cos(100\pi t) + \frac{2}{200\pi}\cos(200\pi t).$$





Hence, in $M_2(\omega)$ the low frequency component amplitude is emphasized more over the high frequency component amplitude than is the case in $M_1(\omega)$. This means that $m_2(t)$ has a smoother time-domain behavior than $m_1(t)$. Thus, the maximum slope of $m_2(t)$ is smaller than the maximum slope of $m_1(t)$.

Now, using the condition $\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right|$, we can conclude that signal $m_2(t)$ requires a smaller Δ to avoid slope overload distortion.

The following alternative procedure can also be used:

$$\left|\frac{dm_1(t)}{dt}\right| = \left|400\pi \sin(100\pi t) + 400\pi \sin(200\pi t)\right|$$

$$\left|\frac{dm_2(t)}{dt}\right| = \left|4 \sin(100\pi t) + 2 \sin(200\pi t)\right|$$

Obviously, the maximum value of $|dm_2(t)/dt|$ is smaller than the maximum value of $|dm_1(t)/dt|$, and $m_2(t)$ requires a smaller Δ to avoid slope overload distortion.

Problem #5

$$(SNR)_{PCM} = \frac{\sigma_M^2}{\frac{\Delta_{PCM}^2}{12}} = \frac{\sigma_M^2}{\left(\frac{DR}{L}\right)^2/12}$$

DR is the dynamic range of the quantizer input, L is the number of quant. levels

$$(SNR)_{DPCM} = \frac{\sigma_M^2}{\frac{\Delta_{DPCM}^2}{12}} = \frac{\sigma_M^2}{\left(\frac{DR/2}{L}\right)^2/12}$$

$$= 4\,(SNR)_{PCM}$$

$\Rightarrow$ Signal-to-noise ratio improvement factor is 4.

Problem #6

$$L_{PCM} = \frac{DR}{\Delta}, \quad L_{DPCM} = \frac{DR/2}{\Delta} = \frac{L_{PCM}}{2}$$

Let R be the number of bits per sample in PCM.

$$\Rightarrow L_{PCM} = 2^R \text{ and } L_{DPCM} = \frac{2^R}{2} = 2^{R-1}$$

Hence, the bit rate in DPCM is:

$$BR_{DPCM} = \frac{R-1}{T_s} = \frac{R}{T_s} - \frac{1}{T_s} = BR_{PCM} - \frac{1}{T_s}$$

$$\Rightarrow R_2 = \frac{1}{T_s}.$$

For a voice signal of 4KHz bandwidth and sampled at the Nyquist rate with 256 PCM levels, the bit rate is reduced from 64,000 bits/sec to 56,000 bits/sec.

Problem #7

$$(SNR)_{PCM} = \frac{\sigma_M^2}{\left(\frac{DR}{L_{PCM}}\right)^2/12}$$

desired equality

$$(SNR)_{DPCM} = \frac{\sigma_M^2}{\left(\frac{DR/4}{L_{DPCM}}\right)^2/12} \stackrel{!}{=} 4\frac{\sigma_M^2}{\left(\frac{DR}{L_{PCM}}\right)^2/12}$$

$$\Rightarrow L_{DPCM} = L_{PCM}/2$$

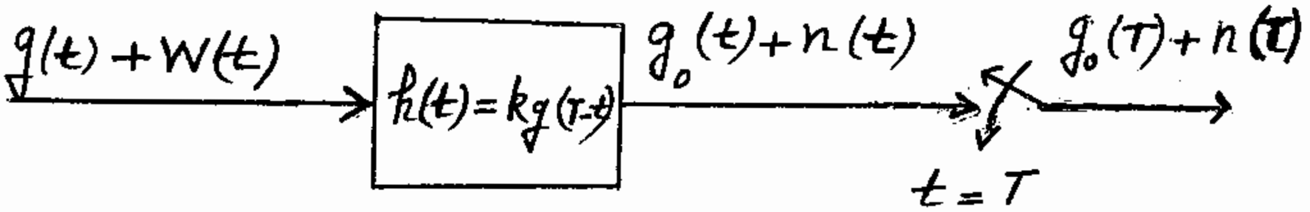
$$\Rightarrow BR_{DPCM} = BR_{PCM} - \frac{1}{T_s} \Rightarrow R_2 = \frac{1}{T_s}.$$

In this problem DPCM is capable of achieving a SNR improvement and bit rate reduction over PCM.

Problem #8

A possible way by which the predictor coefficients in ADPCM at the receiver can be obtained consists of encoding and transmitting on-line the updated predictor coefficients at the transmitter. These encoded coefficients are to be decoded and used as coefficients of the predictor used at the receiver. This procedure results, of course, in an increase in the transmission bit rate over that obtained in DPCM. This is a disadvantage.

Problem # 9



$$g(t) = rect\left[\frac{(t - T/2)}{T}\right]$$

The matched filter maximizes the output $(SNR)_o$ expressed as: $(SNR)_o = \dfrac{|g_o(T)|^2}{E[n^2(t)]}$ and makes this ratio equal to $\dfrac{2E}{N_o}$, where E is the energy of the pulse $g(t)$.

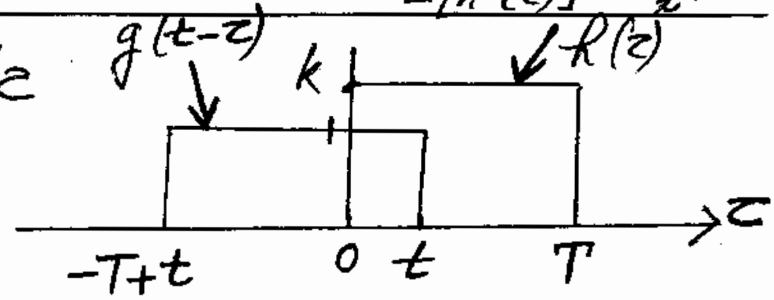
Hence, with $E = \int_0^T (1)^2 dt = T$ and $g_o(T) = kT$, then the following equality holds:

$$\frac{[kT]^2}{E[n^2(t)]} = \frac{2T}{N_0} \Rightarrow E[n^2(t)] = \frac{k^2 T N_o}{2}.$$

If $k = \frac{1}{\sqrt{T}}$ in the case where $h(t)$ is normalized, then $E[n^2(t)] = \frac{N_o}{2}$.

Remarks

1) $g_o(t) = \int_{-\infty}^{\infty} h(\tau) g(t-\tau) d\tau$

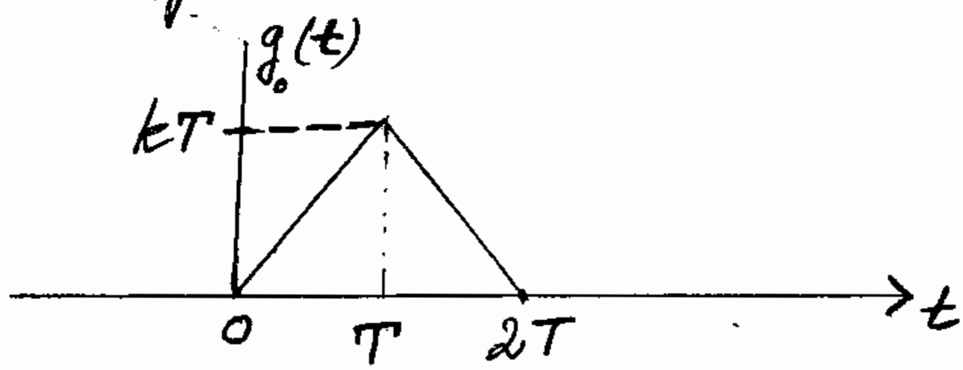


$g_o(t) = 0$, for $t < 0$

$g_o(t) = \int_0^t k\,d\tau = kt, \quad 0 \le t \le T$

$g_o(t) = \int_{-T+t}^{T} k\,d\tau = -kt + 2kT, \quad T \leqslant t \leqslant 2T$

$g_0(t) = 0$, for $t \geq 2T$.



2) $E[n^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} |H(\omega)|^2 d\omega$

$h(t) = kg(T-t) = kg(t)$ for the giving $g(t)$.

$\Rightarrow H(\omega) = k\, G(\omega)$.

$$\Rightarrow E[n^2(t)] = \frac{N_0}{2} \times k^2 \times \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega}_{=E}$$

$$= \frac{k^2 N_0}{2} \times E = \frac{k^2 T N_0}{2}$$ (Same as before).

3) $H(\omega) = k\, G(\omega) = k\int_0^T e^{-j\omega t} d\omega = kTe^{-j\frac{\omega T}{2}} Sa\left(\frac{\omega T}{2}\right)$

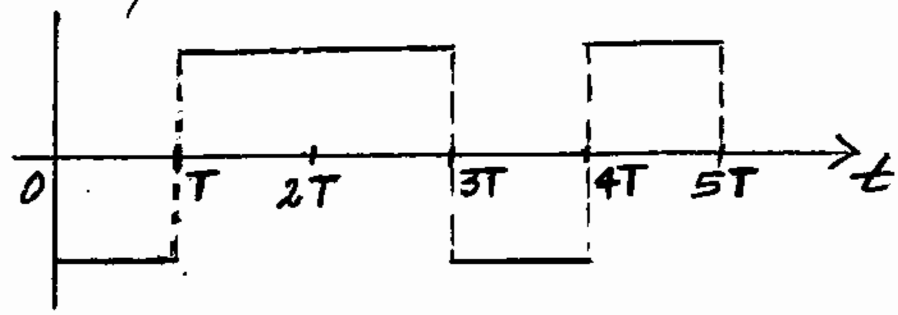
$$\Rightarrow E[n^2(t)] = \frac{N_0}{2} k^2 \times \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$= \frac{N_0}{2} k^2 \times \frac{1}{2\pi}\int_{-\infty}^{\infty} T^2 Sa^2\left(\frac{\omega T}{2}\right) d\omega = \frac{k^2 T N_0}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} Sa^2\left(\frac{\omega T}{2}\right) = \frac{2\pi}{T}.$$

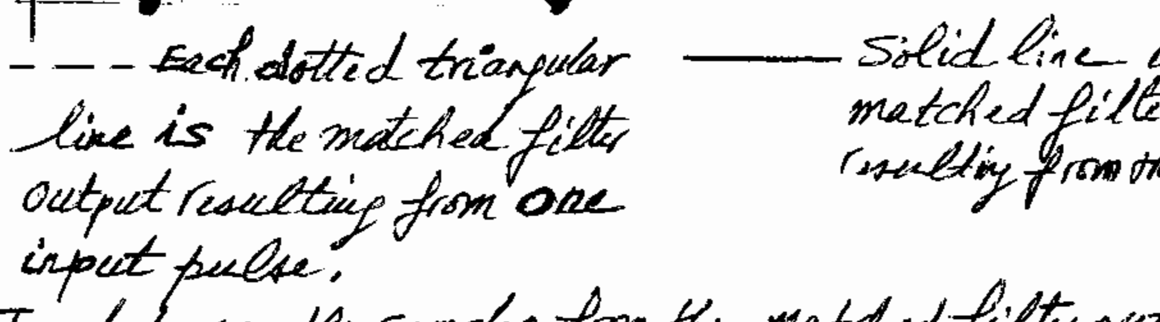
Problem #10

Since the transmission is baseband binary using PNRZ with rectangular pulses, and since the channel is ideal and linear, then the received pulsed sequence at the matched filter input and corresponding to 01101 can be represented as:



The matched filter impulse response is:

The matched filter output is (see remark 1, page 7):



- - - - Each dotted triangular line is the matched filter output resulting from one input pulse.

——— Solid line is the matched filter outp resulting from the sequen

• The dots are the samples from the matched filter output at $t = T, 2T, 3T, 4T$ and $5T$.

Each sample from the matched filter output is only obtained from the output corresponding to a single input pulse and it is not affected by other received pulses. Hence, there is no ISI in the matched filter output.