Quiz #1 Solution Fall2002/0

Question 1

(a) $F_X(x) = P_r\{X \leq x\}$.

For $x < 10$, $P_r\{X \leq x\} = 0$

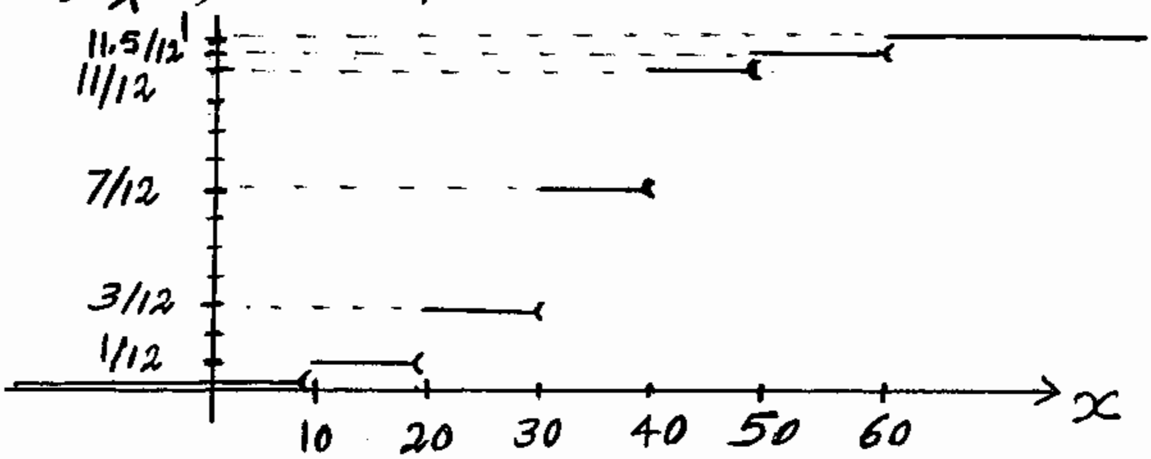
$10 \leq x < 20$, $P_r\{X \leq x\} = P_r\{1\} = \frac{1}{12}$

$20 \leq x < 30$, $P_r\{X \leq x\} = P_r(\{1\} \cup \{2\}) = P_r\{1\} + P_r\{2\}$

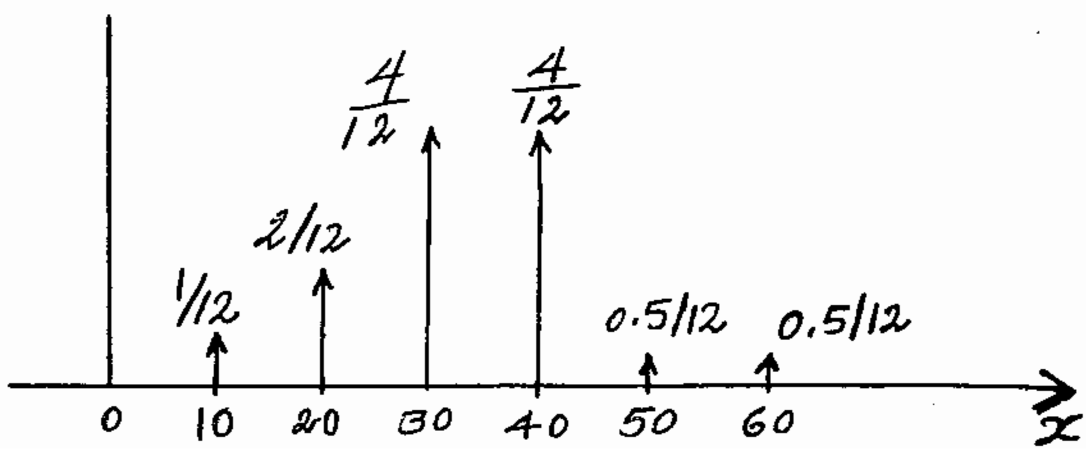
$= \frac{1}{12} + \frac{1}{6} = \frac{3}{12}$.

⋮

Hence, $F_X(x)$ is as plotted below.



$f_X(x) = \frac{dF_X(x)}{dx}$.



(b) $A = \{2,3\} \cup \{2,4,5\} = \{2,3,4,5\}$

$$P_r(A) = P_r\{2\} + P_r\{3\} + P_r\{4\} + P_r\{5\}$$

$$= \frac{2}{12} + \frac{4}{12} + \frac{4}{12} + \frac{0.5}{12} = \frac{10.5}{12} = \frac{21}{24}.$$

Also, $P_r(A) = P_r\{20^- < X \leqslant 50\}$

$$= F_X(50) - F_X(20^-) = \frac{11.5}{12} - \frac{1}{12}$$

$$= \frac{10.5}{12} = \frac{21}{24} = \frac{7}{8}$$

Note that $P_r(A)$ can also be written as:

$P_r(A) = P_r\{m < X \leqslant 50\}$, where $m \in [10, 20)$

$$= F_X(50) - F_X(m) = \frac{21}{24}.$$

Now, using $f_X(x)$:

$$P_r(A) = \int_{20}^{50} f_X(x)\,dx = \frac{2}{12} + 2\times\frac{4}{12} + \frac{0.5}{12}$$

$$= \frac{10.5}{12} = \frac{21}{24}$$

$$= \int_{m^+}^{50} f_X(x)\,dx = \frac{21}{24}.$$

__Question 2__ $X_1 = \int_0^T X(t)\phi_1(t)\,dt$

$$X_2 = \int_0^T X(t)\phi_2(t)\,dt$$

(a)

$$E(X_1) = \int_0^T E[X(t)]\phi_1(t)\,dt = 0\text{; Also, } E(X_2) = 0.$$

$$Var(X_1) = E[X_1^2] = E\left[\int_0^T X(t)\phi_1(t)\,dt \int_0^T X(u)\phi_1(u)\,du\right]$$

$$= \int_0^T \phi_1(t)\,dt \int_0^T \underbrace{E[X(t)X(u)]}_{\frac{N_0}{2}\delta(t-u)} \phi_1(u)\,du$$

$$= \frac{N_0}{2}\int_0^T \phi_1^2(t)\,dt = \frac{N_0}{2}$$

Similarly, $Var(X_2) = \frac{N_0}{2}$.

$$Cov(X_1, X_2) = E[X_1 X_2]$$

$$= E\left[\int_0^T X(t)\phi_1(t)\,dt \int_0^T X(u)\phi_2(u)\,du\right]$$

$$= \int_0^T \phi_1(t)\,dt \int_0^T \underbrace{E[X(t)X(u)]}_{\frac{N_0}{2}\delta(t-u)} \phi_2(u)\,du$$

$$= \int_0^T \frac{N_0}{2}\phi_1(t)\phi_2(t)\,dt = 0$$

(b) X_1 is $N\left(0, \frac{N_0}{2}\right)$

X_2 is $N\left(0, \frac{N_0}{2}\right)$

X_1 and X_2 are also uncorrelated and since they are Gaussian random variables, then they are also statistically independent. Hence,

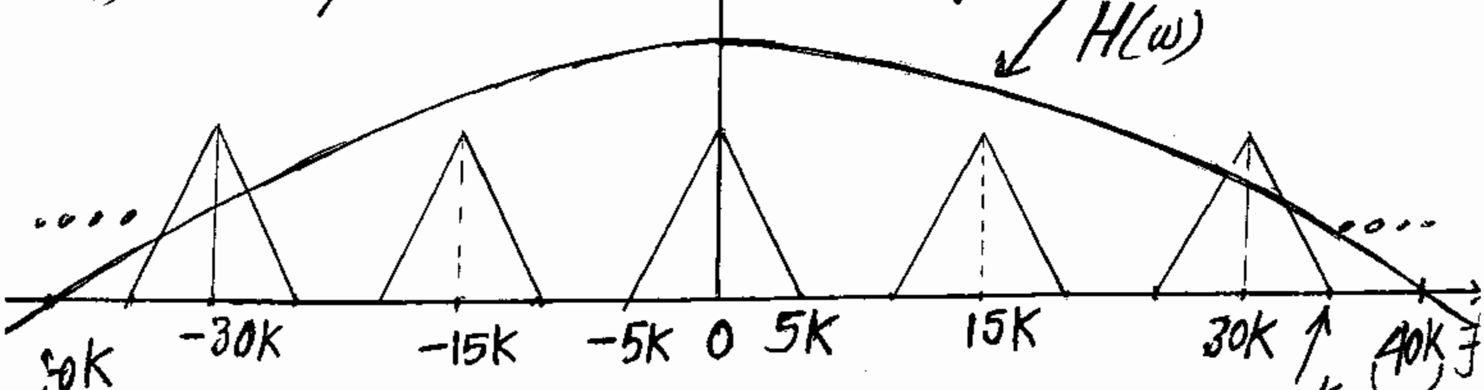
$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2)$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0} x_1^2\right\} \times \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0} x\right.$$

$$= \frac{1}{\pi N_0} \exp\left\{-\frac{(x_1^2 + x_2^2)}{N_0}\right\}$$

Question 3

(a) The spectrum of the PAM signal is:



$B_T = 35\,KHz$

(b) Let T be the duration of $h(t)$.

The largest T is such that $\frac{1}{T} = 50\,KHz$

$$\Rightarrow T = \frac{1}{50\times10^3} = 20\times10^{-6}\ secs = 20\,\mu sec.$$

Question 4

$$m(t) = A\sin(\omega_m t) = 2\sin(\pi\times10^4 t)$$

(a)

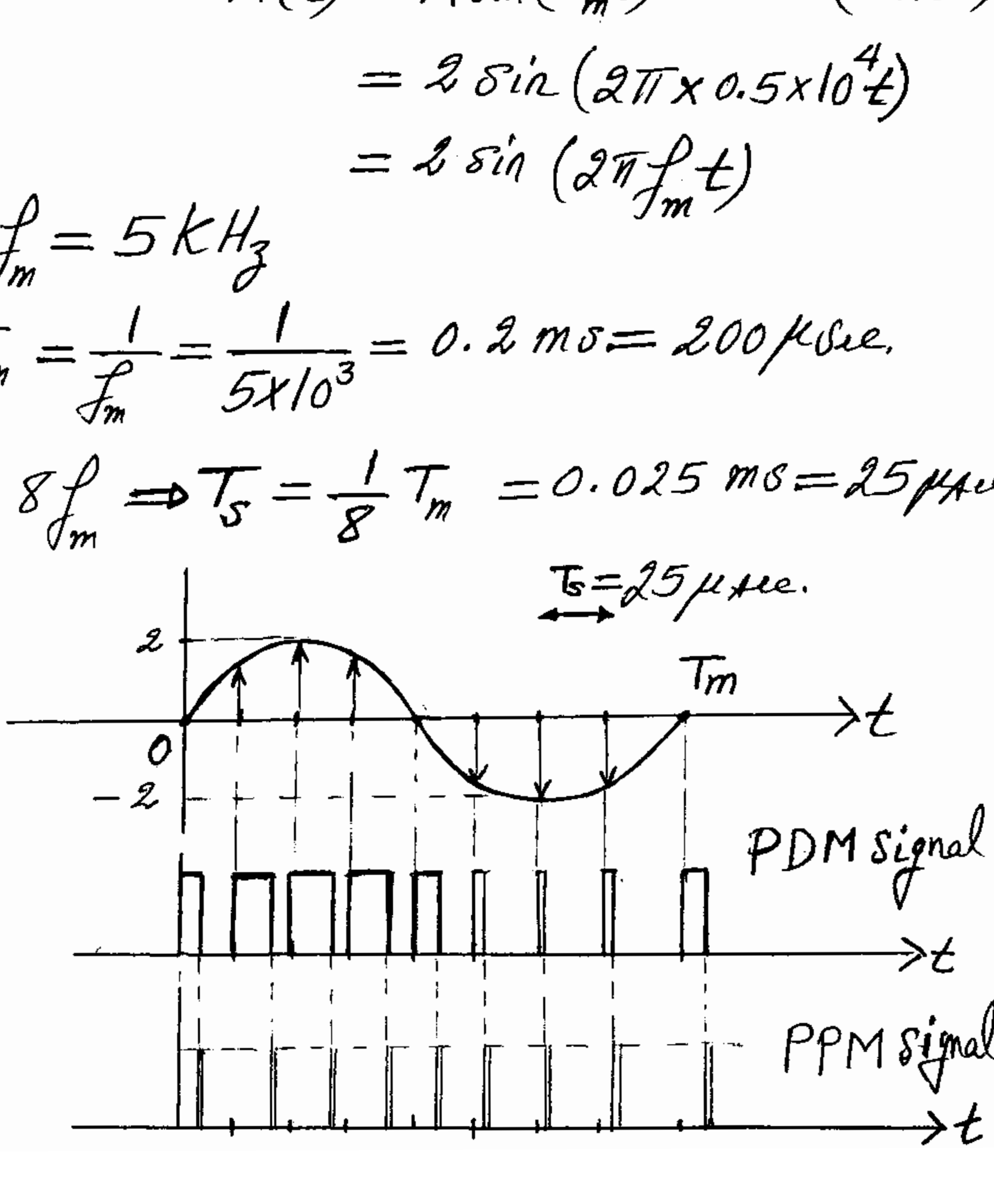
$$= 2\sin(2\pi\times0.5\times10^4 t)$$

$$= 2\sin(2\pi f_m t)$$

$$\Rightarrow f_m = 5\,KHz$$

$$T_m = \frac{1}{f_m} = \frac{1}{5\times10^3} = 0.2\,ms = 200\,\mu sec.$$

$$f_s = 8 f_m \Rightarrow T_s = \frac{1}{8}T_m = 0.025\,ms = 25\,\mu sec$$



The width of the PDM pulses is calculated as follows using: $D = \frac{m(kT_s)}{24\times10^4} + 0.1\times10^{-4}$ secs.

$$m(T_s) = m(3T_s) = 2\sin(\pi\times10^4\times0.025\times10^{-3})$$
$$= 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2} = 1.4142$$

$$m(5T_s) = m(7T_s) = -1.4142.$$

$$m(2T_s) = 2,\ m(6T_s) = -2$$

$$\Rightarrow D(0) = D(4T_s) = D(8T_s) = 0.01\ ms = 10\ \mu sec$$

$$D(T_s) = D(3T_s) = 0.01589\ ms = 15.89\ \mu sec.$$

$$D(2T_s) = 0.01833\ ms = 18.33\ \mu sec.$$

$$D(5T_s) = D(7T_s) = 0.00411\ ms = 4.11\ \mu sec.$$

$$D(6T_s) = 0.00167\ ms = 1.67\ \mu sec.$$

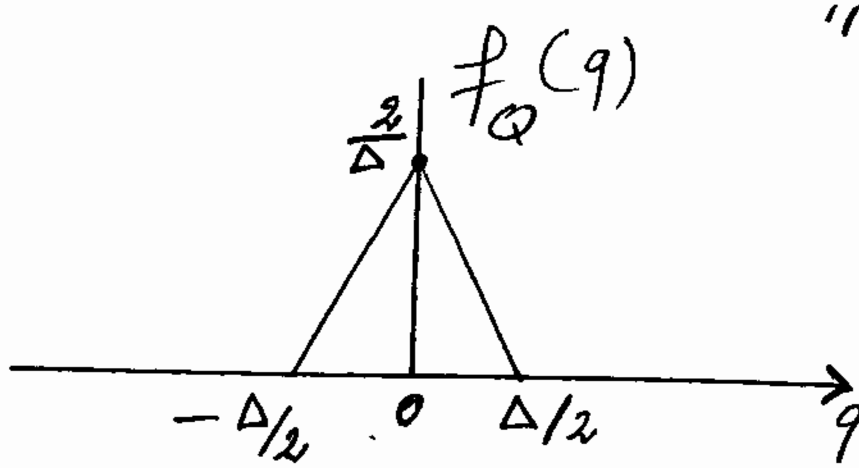
(b) The bandwidth of the PDM signal is obtained using the pulse with the shortest duration:

$$B_T = \frac{1}{1.67\times10^{-6}} = 598.8\ KHz$$

For the PPM signal: $B_T = \frac{1}{10^{-6}} = 1\ MHz.$

'11'

Question 5



(a)

$q = m - v$ where m is the sample value from the analog message and v is the quantized value of the sample.

The above $f_Q(q)$ implies that $|m - v|$'s occur more frequently near the zero value than near the $\frac{\Delta}{2}$ value. Hence, the majority of the sample values m occur near the v's (quantization levels) than near the $\left(v \pm \frac{\Delta}{2}\right)$'s, which are the cell boundaries or decision thresholds.

(b)
$$f_Q(q) = \begin{cases} \frac{4}{\Delta^2} q + \frac{2}{\Delta}, & -\frac{\Delta}{2} \leq q \leq 0 \\ -\frac{4}{\Delta^2} q + \frac{2}{\Delta}, & 0 \leq q \leq \frac{\Delta}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$E(Q^2) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 f_Q(q)\, dq = 2\int_0^{\frac{\Delta}{2}} q^2 f_Q(q)\, d$$

$$= 2\int_0^{\frac{\Delta}{2}} q^2 \left[-\frac{4}{\Delta^2} q + \frac{2}{\Delta}\right] dq$$

$$= 2\left[-\frac{1}{\Delta^2} q^4 \Big|_0^{\frac{\Delta}{2}} + \frac{2}{3\Delta} q^3 \Big|_0^{\frac{\Delta}{2}}\right]$$

$$= 2\left[-\frac{\Delta^2}{16} + \frac{\Delta^2}{12}\right] = \Delta^2/24.$$

The quantization noise can be reduced by reducing Δ or equivalently increasing the number of quantizat. levels.

The principle of bandwidth-noise tradeoff relies on the quantizer output signal-to-noise ratio.

$$(SNR)_0 = \frac{\sigma_M^2}{\sigma_Q^2} = \frac{\sigma_M^2}{\frac{\Delta^2}{24}} = \frac{24\sigma_M^2}{\left(\frac{2m_{max}}{L}\right)^2} = \frac{6\sigma_M^2 L^2}{m_{max}^2}$$

$$= \frac{6\sigma_M^2 \times 2^{2R}}{m_{max}^2}$$. But R is % to the transmissi. bandwidth. $\Rightarrow$ Principle is holding

Question 6

(a) Since uniform quantization uses a constant separation between the quantization levels, then the higher amplitude values of the message need to be compressed more than the lower amplitude values in order that uniform quantization applied to the compressed signal becomes equivalent to non-uniform quantization as described.

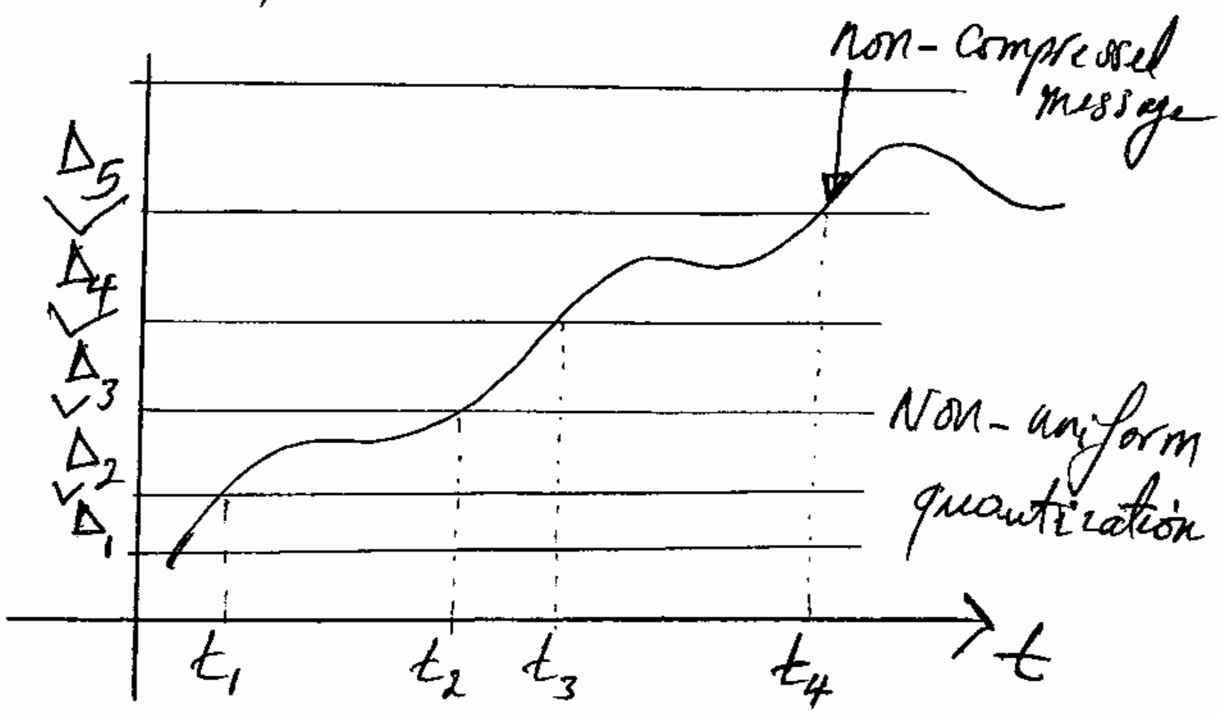


Figure 1

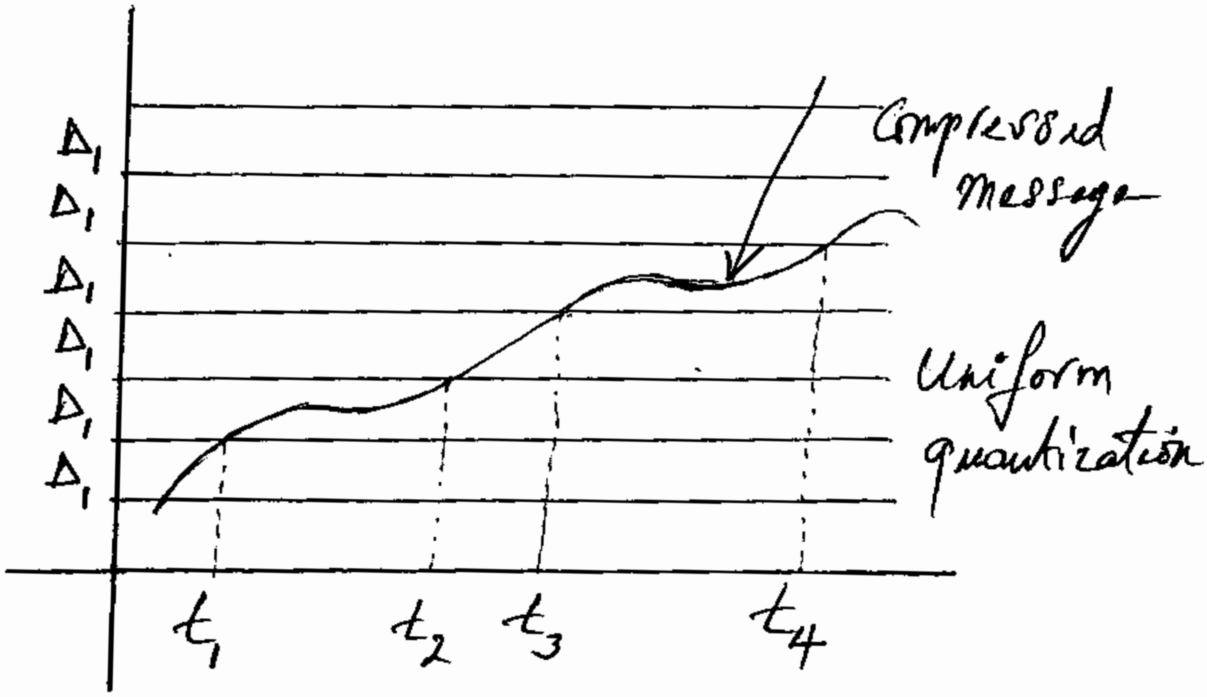


Figure 2.

In order that the signal values in Figure 1 between t_2 and t_3 (and which are in Δ_3) and the values between t_1 and t_2 (and which are in Δ_2), with $\Delta_3 > \Delta_2$, be accommodated within $\Delta_1 < \Delta_2 < \Delta_3$, the values in Δ_3 need to be compressed more than the values in Δ_2. Also, values in Δ_4 need to be compressed more than the values in Δ_3 and Δ_2.

(b) From the plot of the μ-law for $\mu = 100$, the following can be obtained:

$$|m_1| = 0.2 \longrightarrow |m_{01}| \cong 0.8$$

$$|m_2| = 0.6 \longrightarrow |m_{02}| \cong 0.9$$

But $|m| = \left|\dfrac{V}{V_{max}}\right|$, and $|m_0| = \left|\dfrac{V_c}{V_{c\,max}}\right|$

Let us consider positive values only.

$$\Longrightarrow \left.\begin{array}{l} V_1 = 0.2\ V_{max} \\ V_{c_1} = 0.8\ V_{c\,max} \end{array}\right\} \Longrightarrow \frac{V_{c_1}}{V_1} = 4\left(V_{c\,max}/V_{max}\right)$$

$$\left.\begin{array}{l} V_2 = 0.6\ V_{max} \\ V_{c2} = 0.9\ V_{c\,max} \end{array}\right\} \Longrightarrow \frac{V_{c2}}{V_2} = 1.5\left(V_{c\,max}/V_{max}\right)$$

Now, since $V_{c\,max} < V_{max} \Rightarrow 0 < \dfrac{V_{c\,max}}{V_{max}} = \alpha < 1.$

Let $\alpha = 0.2 \Rightarrow$

$$\Longrightarrow V_{c_1} = 4\alpha\, V_1 = 0.8\, V_1$$

$$V_{c2} = 1.5\alpha\, V_2 = 0.3\, V_2$$

Hence $V_2 > V_1$ and V_2 is compressed more than V_1.

Question 7 $\quad E_{br} = \dfrac{K}{d^2} E_{bt}$; $K = 9 \times 10^{-2}\, m^2$

$$N_0 = 2 \times 10^{-12}\ V^2 \cdot sec$$

For an acceptable regeneration of the sequence of digits at the first repeater, given the use of PNRZ signaling and a bit rate equal to 64000 bits/sec, the signal-to-noise ratio at the first repeater has to be greater than or equal to 11 dB. The SNR is equal to (E_{br}/N_0).

Hence, $10 \log_{10}\left(\dfrac{E_{br}}{N_0}\right) = 10 \log_{10}\left[\dfrac{K E_{bt}}{N_0 d^2}\right] \geqslant 11\,dB$.

With $E_{bt} = A^2 T_b = \dfrac{A^2}{R_b}$, then

$$10 \log_{10}\left[\frac{K A^2}{N_0 R_b d^2}\right] \geqslant 11\,dB$$

$$\Rightarrow \frac{K A^2}{N_0 R_b d^2} \geqslant 10^{1.1} \Rightarrow d^2 \leqslant \frac{K A^2}{10^{1.1} N_0 R_b} = \frac{9 \times 10^{-2} \times 64}{10^{1.1} \times 2 \times 10^{-12} \times 64 \times 1\!\!(}$$

$$= \frac{9 \times 10^{7}}{2 \times 10^{1.1}}$$

$$= 4.5 \times 10^{5.9}$$

$\Rightarrow d \leqslant 1.89\ Km.$

Question 8

$$m(t) = 2\cos(2\pi \times 10^3 t) \text{ volts.}$$

a)

a.1 - $\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right| = (2 \times 2\pi \times 10^3)$

$= 4\pi \times 10^3$

$$\Rightarrow \Delta \geq \frac{4\pi \times 10^3}{f_s} = \frac{4\pi 10^3}{4 \times 2 \times 10^3} = \frac{\pi}{2} \text{ Volts.}$$

The smallest Δ is $\frac{\pi}{2}$ Volts.

a.2 - $\frac{\Delta}{T_s} \geq \max\left|\frac{d}{dt}\int m(\tau)d\tau\right] = \max(m(t))$

$= 2$

$$\Rightarrow \Delta \geq \frac{2}{f_s} = \frac{2}{8 \times 10^3} = 0.25 \times 10^{-3} \text{ Volts}$$

$= 0.25$ mV.

The smallest Δ is 0.25 mV.

(b) The signal that is less sloppy is the integrated version of $m(t)$. Since the Δ needed to avoid slope overload distortion in $m(t)$ is much larger than the needed Δ in the integral of $m(t)$.
A higher Δ is needed, in fact, when the signal variation is fast to make $m_q(t)$ approach closely $m(t)$

Question 8

$$\Delta_{PCM} = \frac{2m_{max}}{L} \Rightarrow (SNR)_{PCM} = \frac{P_M}{\frac{\Delta_{PCM}^2}{12}}$$

$$= \frac{12 P_M L^2}{4m_{max}^2} = \frac{3 P_M L^2}{m_{max}^2}$$

$$\Delta_{DPCM} = \frac{\frac{2m_{max}}{20}}{L} \Rightarrow (SNR)_{DPCM} = \frac{P_M}{\frac{\Delta_{DPCM}^2}{12}}$$

$$= \frac{12 P_M L^2}{4 m_{max}^2} \times 400$$

$$= \frac{3P_M L^2}{m_{max}^2} \times 400$$

$$= 400 \times (SNR)_{PCM}$$

Hence, $10\log_{10}(SNR)_{DPCM} = 10\log_{10}(SNR)_{PCM} + 10\log_{10} 4\times10^2$

$$\Rightarrow (SNR)_{DPCM}\, dB = (SNR)_{PCM}\, dB + 26.02$$