Quiz #1 Solution — Fall 2003-0?

Problem #1

$$s(t) = A\cos(\omega_c t), \quad 0 \leqslant t \leqslant T$$

$s(t)$ can be written as follows:

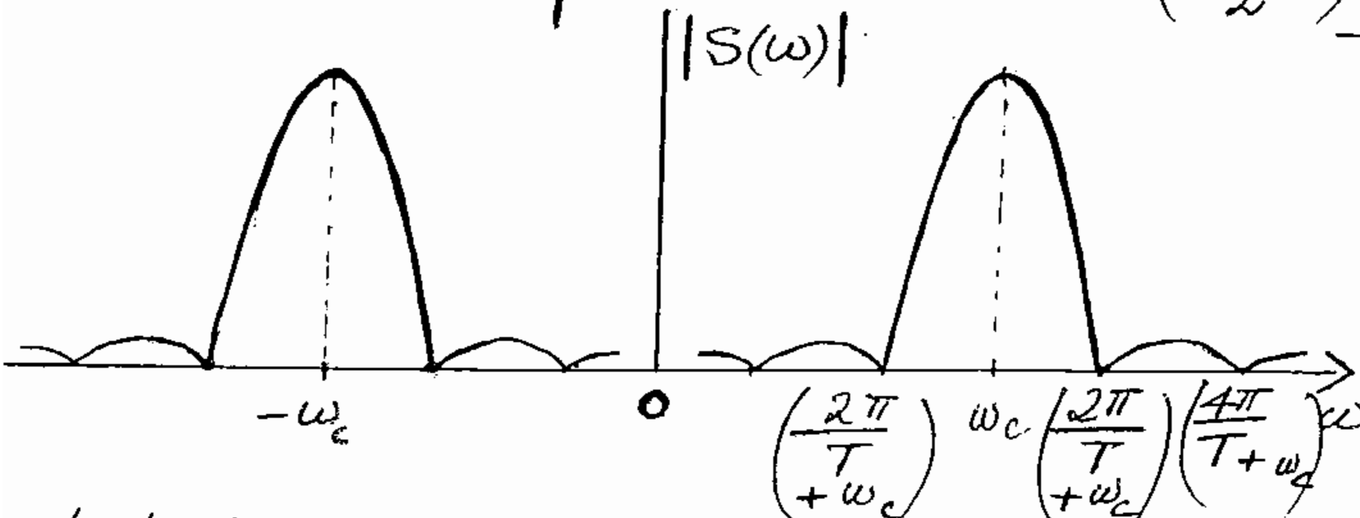
$$s(t) = A\cos(\omega_c t)\,\mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

$$\Rightarrow S(\omega) = \mathcal{F}\{A\cos(\omega_c t)\} * \mathcal{F}\left\{\mathrm{rect}\left(\frac{t - T/2}{T}\right)\right\} \times \frac{1}{2\pi}$$

$\Rightarrow$

$$S(\omega) = \frac{T e^{-j\frac{\omega T}{2}}}{2\pi}\left[\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right] * \left[A\pi\delta(\omega-\omega_c) + A\pi\delta(\omega+\omega_c)\right]$$

$$|S(\omega)| = \frac{AT}{2}\left|\left[S_a\left(\frac{(\omega-\omega_c)T}{2}\right) + S_a\left(\frac{(\omega+\omega_c)T}{2}\right)\right]\right|$$



The bandwidth of $s(t)$ is:

$$W = 2\pi B = \frac{4\pi}{T}\ \text{rad/s}$$

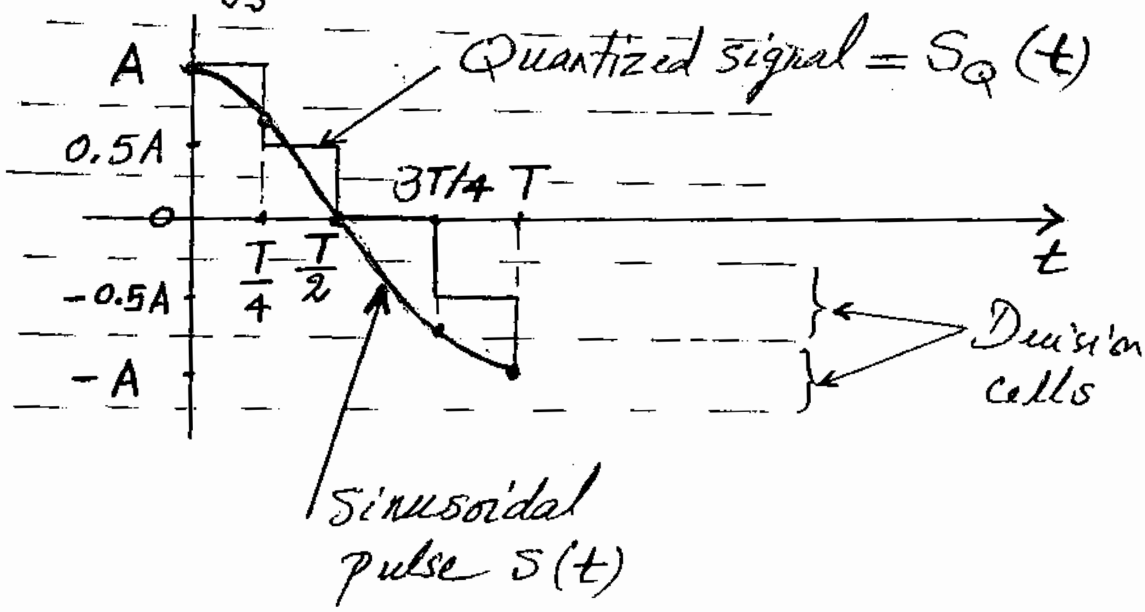
$$\Rightarrow B = \frac{2}{T}\ \text{Hz.}$$

4 (a) $f_s = 2B = \frac{4}{T}$ samples/sec.

4 (b) $s_p(t) = A\cos(\omega_c t) = A\cos\left(\frac{\pi}{T}t\right)$

is periodic with period equal to $2T$.

$T_s = \frac{1}{f_s} = \frac{T}{4}$



The energy of the quantized signal is:

$$E_{SQ} = \int_0^{T/4} A^2 dt + \int_{T/4}^{T/2} (0.5A)^2 dt + \int_{T/2}^{3T/4} 0\, dt + \int_{3T/4}^{T} (-0.5A)^2 dt$$

$$= \frac{A^2 T}{4} + 2\times(0.5A)^2 \frac{T}{4} = \frac{1.5A^2T}{4} = 0.375A^2T$$

4 (c) $$E_s = \int_0^T A^2\cos^2(\omega_c t)\,dt = \frac{A^2}{2}\left[\int_0^T dt + \int_0^T \cos(2\omega_c t)\,dt\right]$$

$$= 0.5A^2T$$

The quantizer output signal-to-noise ratio is:

$$(SNR)_{OQ} = \frac{0.5A^2T}{0.5A^2T - 0.375A^2T} = 4$$

Problem #2

In a binary sequence of duration 40 secs., where the bit duration is 2 μsecs, there are 20,000,000 bits. Now, since the bit error probability is 10^{-6}, then the number of bits which could be detected in error in the noted sequence is 20.

4 (a) The maximum number of codewords that could be decoded to the wrong quantization levels is obtained by considering the 20 bits that detected in error distributed over 20 different codewords. It is sufficient to have one bit error in a codeword to have the codeword decoded to the wrong quantization level. So, the answer is 20 codewords.

4 (b) The minimum number of codewords that could be decoded to the wrong levels is obtained by assuming that the 20 error bits occur in 3 different codewords at 8, 8, 4. So, the answer is 3 codewords.

Problem #3

4 (a) The bit rate in the DM system is:

$8 \times 2 \times 4000 = 64{,}000$ bits/sec = sampling rate

The bit rate in the TDM-DM system is:

$20 \times 64{,}000 = 1{,}280{,}000$ bits/sec.

$\Rightarrow B_T = 1.28\ MHz.$

Or, $T_s = \dfrac{1}{64{,}000}$ secs.

$$T_b = \frac{T_s}{20} = \frac{1}{20 \times 64{,}000} = \frac{1}{1{,}280{,}000} \text{ secs.}$$

$$\Rightarrow B_T = \frac{1}{T_b} = 1.28\ MHz.$$

4 (b) Let f_s be the sampling rate in PCM.

Then, the bit rate in the TDM-PCM system is:

$20 \times 8 \times f_s$.

Hence, $20 \times 8 \times f_s = 1{,}280{,}000$ bits/sec.

$$\Rightarrow f_s = \frac{1{,}280{,}000}{20 \times 8} = 8000 \text{ samples/sec.}$$

$= 2 \times 4\ KHz$

$=$ Nyquist rate of sampling

Problem #4

$$r(nT_b) = \begin{cases} A + W(nT_b), & \text{if bit 1 is transmitted} \\ W(nT_b), & \text{if bit 0 is transmitted.} \end{cases}$$

If $r(nT_b) > A/2$, decide 1

If $r(nT_b) < A/2$, decide 0.

(a) $\text{Prob}(\text{decide 0 given 1 is transmitted}) = p_1$

$$= \text{Prob}\left(r(nT_b) < \frac{A}{2} \,/\, r(nT_b) = A + W(nT_b)\right)$$

$$= \text{Prob}\left(A + W(nT_b) < A/2\right)$$

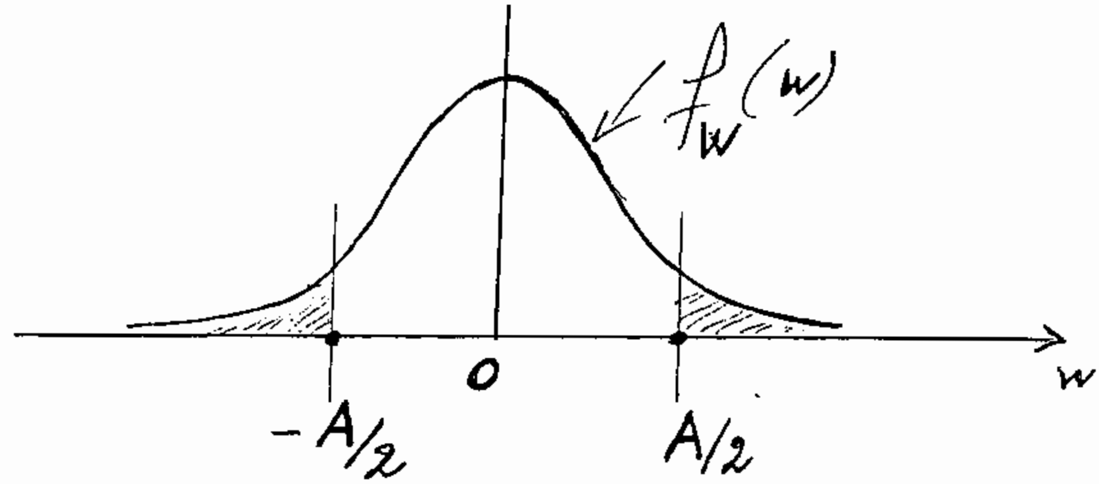
$$= \text{Prob}\left(W(nT_b) < -A/2\right)$$

Now, $\text{Prob}(\text{decide 1 given 0 is transmitted})$

$$= \text{Prob}\left(r(nT_b) > A/2 \,/\, r(nT_b) = W(nT_b)\right)$$

$$= \text{Prob}\left(W(nT_b) > A/2\right) = p_2$$

Since $W(t)$ is a zero-mean stationary Gaussian process with average power σ_W^2, then $W(nT_b)$ is a zero-mean Gaussian random variable with variance σ_W^2.



$$f_W(w) = \frac{1}{\sqrt{2\pi}\,\sigma_W} \exp\left\{-\frac{1}{2}\frac{w^2}{\sigma_W^2}\right\}$$

$$p_1 = \int_{-\infty}^{-A/2} f_W(\omega)\,d\omega; \quad p_2 = \int_{\frac{A}{2}}^{\infty} f_W(\omega)\,d\omega$$

By the symmetry of the problem, $p_1 = p_2$.

Hence, the probability of detecting or deciding 0 given 1 is transmitted is equal to the probability of deciding 1 given 0 is transmitted.

(b) With a $BER = 10^{-4}$, the expected number of bits detected in error in a sequence containing 1,000,000 bits is equal to 100. Hence, with $p_1 = p_2$ and given that the number of 1's is equal to the number of zero's in the transmitted sequence, then the expected number of 0's detected as 1's is 50 and equal to the expected number of 1's detected as

6 Problem #5

$$E_{br} = \frac{K}{d^2} E_{bt}$$

The distance between two consecutive repeaters, used in a digital transmission system with bit rate equal to 64,000 bits/sec and PNRZ transmission technique, is such that:

$$10\log_{10}\left(\frac{E_{br}}{N_0}\right) \geq 11\,dB \Leftrightarrow \frac{E_{br}}{N_0} \geq 10^{1.1} = 12.59$$

$$\Rightarrow \frac{K E_{bt}}{N_0 d^2} \geq 12.59$$

$$\Rightarrow d \leq \left[\frac{K E_{bt}}{N_0 \times 12.59}\right]^{1/2}$$

Equality needs to be used to determine the minimum number of repeaters.

$$\Rightarrow d = \left[\frac{2.5\times10^{-4}\times 20}{10^{-10}\times 12.59}\right]^{1/2} = 1992.84\,m. \simeq 2\,Km$$

The minimum number of repeaters is: $\frac{20\,Km}{2\,km} = 10.$

Problem # 6

4 (a) Let f_s be the sampling rate.

The length of the codeword in the uncompressed signal is R_1 such that $L_1 = 16 = 2^{R_1} \Rightarrow R_1 = 4$.

The length of the codeword in the compressed signal is R_2 such that $L_2 = \frac{D.R.C.S}{\Delta} = \frac{D.R.U.C.S}{2\Delta}$

$$= \frac{16}{2} = 3 = 2^{R_2}$$

$\Rightarrow R_2 = 3$. Hence,

$$\frac{\text{Bit rate for the Compressed signal}}{\text{Bit rate for the uncompressed signal}} = \frac{R_2 \times f_s}{R_1 \times f_s}$$

$$= \frac{R_2}{R_1} = \frac{3}{4}$$

4 (b) Let $p_1, p_2, \ldots, p_{16}$ be the probabilities of the 16 levels.

Also, let $l_1, l_2, \ldots, l_{16}$ be the lengths of the 16 codewords associated with the 16 levels.

The average codeword length of the code is:

$\bar{l} = \sum_{i=1}^{16} p_i l_i$. This needs to be equal to $\sum_{j=1}^{4} p_j \times R_2$

where $p_j, j=1,2,3,4$ are the probabilities of the levels used for the compressed signal. But,

$$\sum_{j=1}^{4} p_j = 1 \Rightarrow \sum_{j=1}^{4} p_j \times R_2 = R_2 = 3 .$$

Hence, the equation that needs to be satisfied in order to make the variable length coding scheme equivalent to the compression scheme in part (a) is:

$$\sum_{i=1}^{16} p_i l_i = 3 .$$

This equation provides the same bit rate under the compression and the variable length coding schemes if the same sampling rate is used.