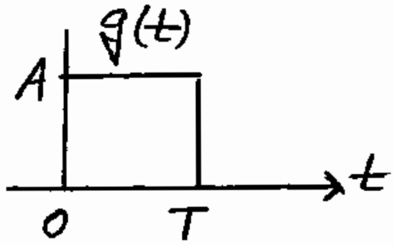
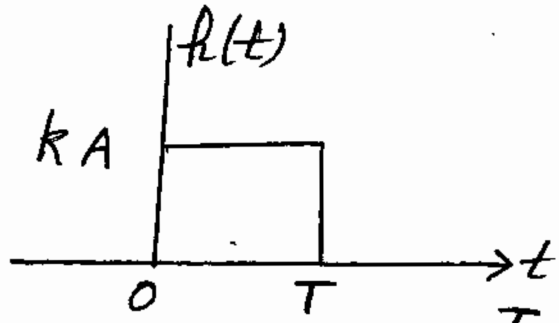
Quiz 2 Solution

Fall 2002-0

Question 1

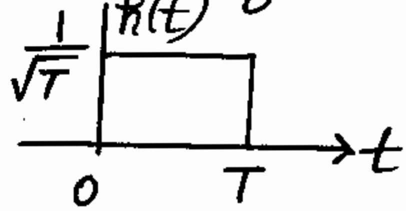




$$h(t) = k\,g(T-t), \text{ where } k \text{ is such that } \int_0^T h^2(t)\,dt = 1.$$

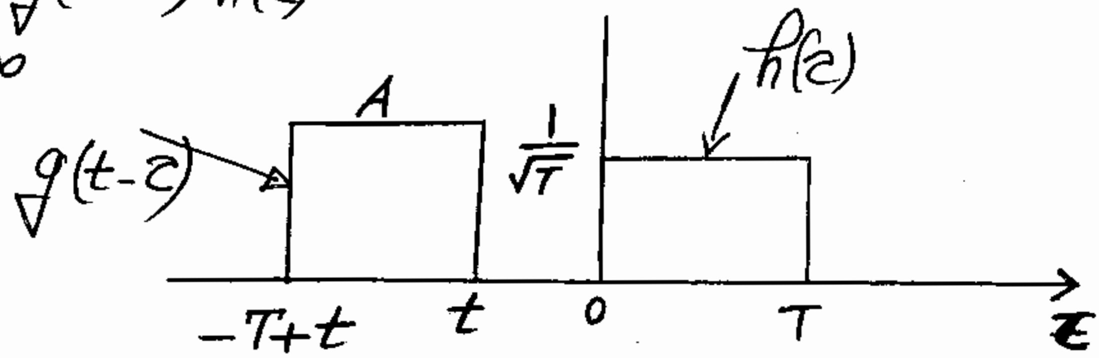
$$\int_0^T h^2(t)\,dt = \int_0^T k^2A^2\,dt = k^2A^2T = 1 \Rightarrow k = \frac{1}{A\sqrt{T}}$$

$\Rightarrow$



The output of the normalized matched filter is:

$$g_0(t) = \int_{-\infty}^{\infty} g(t-\tau)\,h(\tau)\,d\tau$$

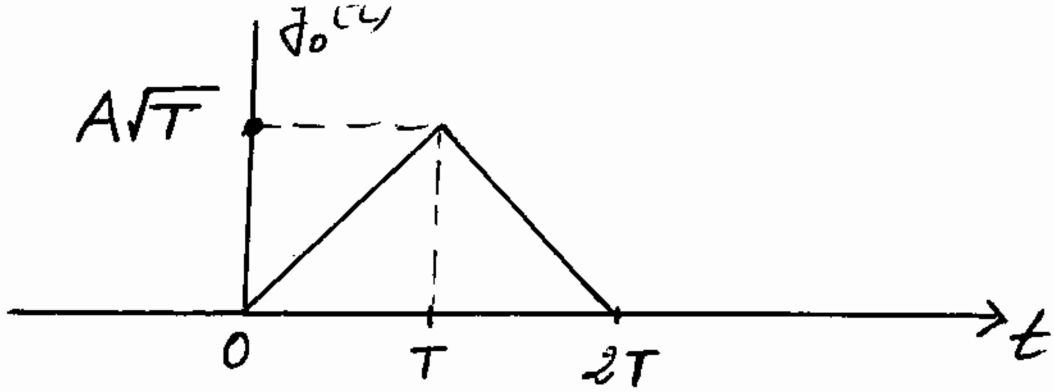


If $t \leq 0$, $g_0(t) = 0$.

$$0 \leq t \leq T,\quad g_0(t) = \int_0^t \frac{A}{\sqrt{T}}\,d\tau = \frac{A}{\sqrt{T}}t$$

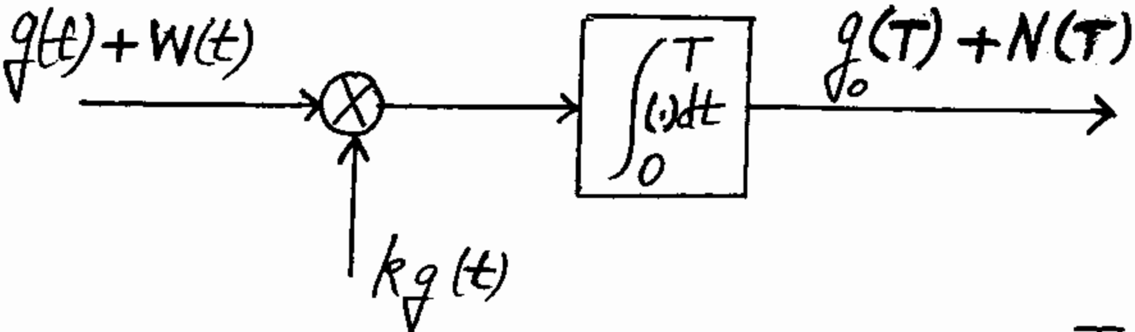
$$T \leq t \leq 2T,\quad g_0(t) = \int_{-T+t}^{T} \frac{A}{\sqrt{T}}\,d\tau = \frac{A}{\sqrt{T}}\tau\Big|_{-T+t}^{T} = \frac{A}{\sqrt{T}}(-t+2T)$$

$t \geq 2T$, $g_0(t) = 0$



Question 2

The matched filter receiver is equivalent to a correlation receiver.



$$N(T) = \int_0^T W(t) \times k\,g(t)\,dt = \frac{1}{A\sqrt{T}} \int_0^T W(t) g(t)\,dt.$$

$$E[N^2(t)] = E[N^2(T)]$$

$$= \frac{1}{A^2 T} E\left[\int_0^T W(t) g(t)\,dt \int_0^T W(u) g(u)\,du\right]$$

$$= \frac{1}{A^2 T} \int_0^T g(t)\,dt \int_0^T \underbrace{E[W(t)W(u)]}_{N_0/2\,\delta(t-u)} g(u)\,du$$

$$= \frac{N_0}{2} \times \frac{1}{A^2 T} \int_0^T g^2(t)\,dt = \frac{N_0}{2}.$$

Question 3

$$Y(T) = Y = \begin{cases} g_0(T) + N(T), & \text{if } 1 \text{ is transmitted} \\ N(T), & \text{if } 0 \text{ is transmitted.} \end{cases}$$

$$= \begin{cases} A\sqrt{T} + N(T), & \text{if } 1 \text{ is transmitted} \\ N(T), & \text{if } 0 \text{ is transmitted.} \end{cases}$$

The random variable $N = N(T)$ is Gaussian with zero mean and variance $\frac{N_0}{2}$. Hence,

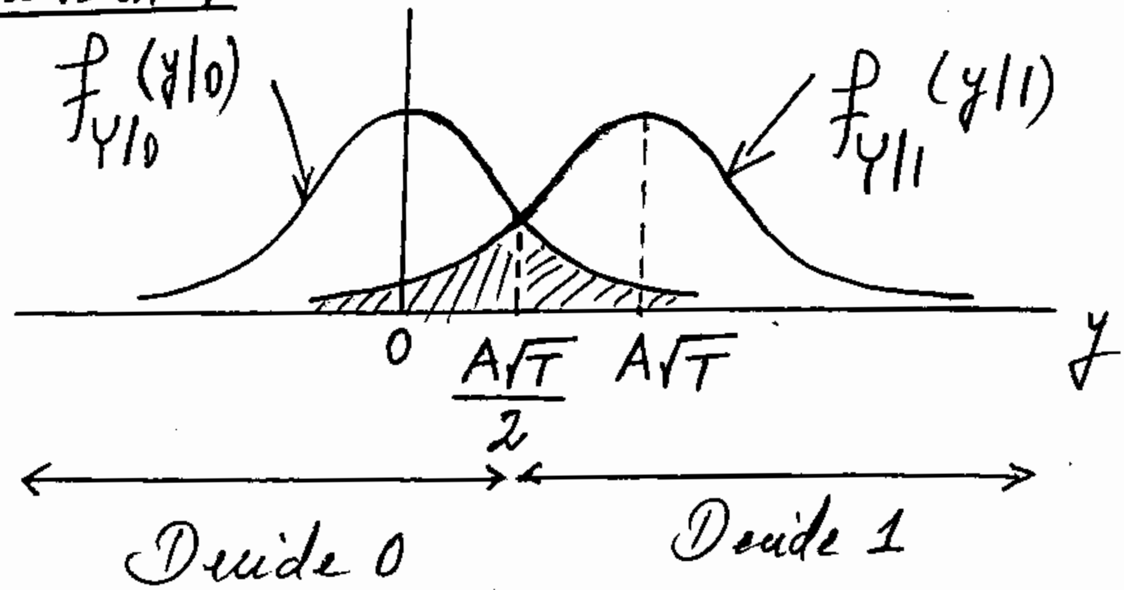
$$f_{Y|1}(y|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0}\left(y - A\sqrt{T}\right)^2\right\}$$

$$f_{Y|0}(y|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0} y^2\right\}$$

Note $g_0(T) = A\sqrt{T}$ can also be determined using the correlation receiver.

$$g_0(T) = \int_0^T k g^2(t)\, dt = \frac{1}{A\sqrt{T}} \times A^2 T = A\sqrt{T}$$

Question 4



$$P_e = \frac{1}{2}\int_{-\infty}^{\frac{A\sqrt{T}}{2}} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0}\left(y - A\sqrt{T}\right)^2\right\}$$

$$+ \frac{1}{2}\int_{\frac{A\sqrt{T}}{2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0} y^2\right\} dy$$

Since the above two integrals are equal, as the shaded areas show, then P_e can be written as:

$$P_e = \int_{-\infty}^{\frac{A\sqrt{T}}{2}} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0}\left(y - A\sqrt{T}\right)^2\right\}$$

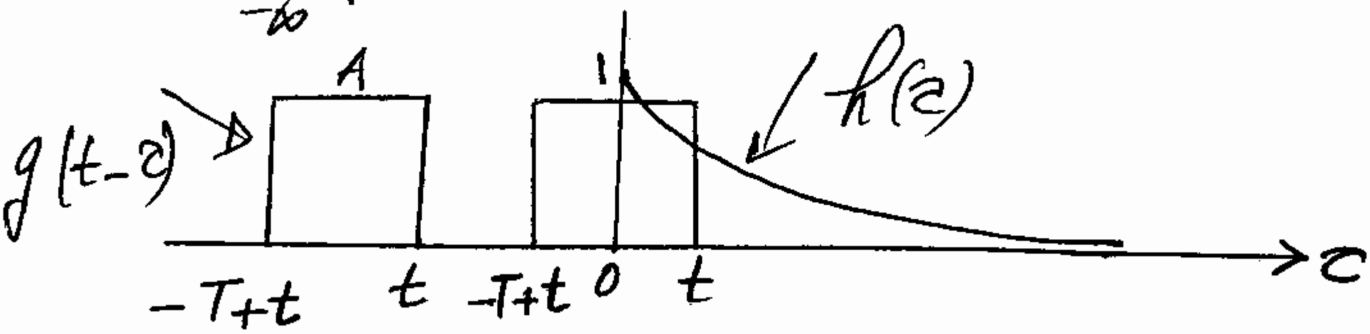
$$= \int_{-\infty}^{\frac{-A\sqrt{T}}{\sqrt{2N_0}}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} z^2\right\} dz$$

$$= \text{erf}_*\left(\frac{-A\sqrt{T}}{\sqrt{2N_0}}\right) = \text{erf}_*\left(-\sqrt{\frac{A^2 T}{2N_0}}\right) = \text{erf}_*\left(-\sqrt{\frac{E}{2N}}\right.$$

Question 5

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow g_0(t)$$

$$g_0(t) = \int_{-\infty}^{\infty} g(t-\tau)\,h(\tau)\,d\tau$$

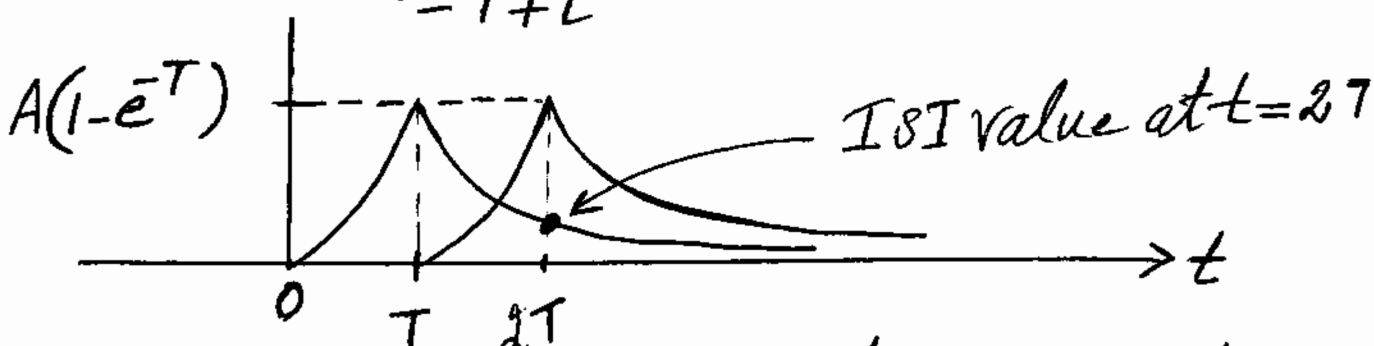


$t \leq 0$, $g_0(t) = 0$

$0 \leq t \leq T$, $g_0(t) = \int_0^t A e^{-\tau} d\tau = -A e^{-\tau}\Big|_0^t$

$$= A(1 - e^{-t})$$

$t \geq T$, $g_0(t) = \int_{-T+t}^{t} A e^{-\tau} d\tau = A(e^{T} - 1)e^{-t}$



Output of channel detector combination for an input represented by the 1 1 sequence.

The ISI value at $t = 2T$ is:

$$A(e^{T}-1)e^{-t}\Big]_{t=2T} = A(e^{-T})(1-e^{-T})$$

Question 6

$$\mu P(\omega) = G(\omega)H(\omega)C(\omega)$$

$$\Rightarrow \mu^2 |P(\omega)|^2 = |G(\omega)|^2 |H(\omega)|^2 |C(\omega)|^2 \quad (1)$$

But: $|G(\omega)|^2 = |C(\omega)|^2 = k\dfrac{|P(\omega)|}{|H(\omega)|}$

Replace in (1), then:

$$\mu^2 |P(\omega)|^2 = K^2 \frac{|P(\omega)|^2}{|\cancel{H}(\omega)|^2} \times |\cancel{H}(\omega)|^2$$

$$= K^2 |P(\omega)|^2$$

So, $P(\omega)$ or $p(t)$ is guaranteed to be obtained at the output of the receiver.