Math 219 — Spring 2015
Quiz I, February 27. TIME: 70 minutes

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

This exam booklet is also your answer sheet; **please answer question 0 on this page, and answer questions 1–6 inside the booklet.** The problems are repeated inside the booklet. There are extra blank sheets at the end for solutions. You can also use the back of any page for solutions or scratchwork. If you need to continue a problem beyond the first page where it appears, please INDICATE CLEARLY where the solution continues.

Read through the problems before starting, and decide which of them you wish to work on first. Do as much of the exam as you can, and budget your time wisely. Each of questions 1–6 counts for 12 points. The exam is closed book, and calculators are not allowed.

Make sure to communicate your ideas by explaining all of your steps precisely and clearly.

Good luck!

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0. a) Your name: Correction b) Your AUB ID#:

1. Let V and W be vector spaces, let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of V, and let $T : V \to W$ be a linear transformation.

a) (3 pts) Carefully define what it means to say that $\operatorname{Span} S = V$.

b) (3 pts) Carefully define the set $\operatorname{Image} T$.

c) (6 pts) Show that if $\operatorname{Span} S = V$, then $\operatorname{Image} T = \operatorname{Span}\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$.

2. a) (8 pts) Show that the set $S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\4\\1\end{pmatrix}\}$ is a basis for $\mathbf{R}^3$.

b) (4 pts) Express the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ as a linear combination of the vectors in the above basis.

3. Define the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\begin{pmatrix}x_1\\x_2\end{pmatrix}) = \begin{pmatrix}x_1 + 2x_2\\2x_2\\x_1\end{pmatrix}$.

a) (3 pts) Find the matrix A_T.

b) (4 pts) Show that T is injective but not surjective.

c) (5 pts) Find **one** linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^2$ such that $S \circ T = id_{\mathbf{R}^2}$ (there are many possible answers), and verify that $A_S A_T = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

4. Given vectors $\vec{x}, \vec{y}, \vec{z} \in V$ with $\{\vec{x}, \vec{y}, \vec{z}\}$ linearly independent.

a) (6 pts) Show that $\{\vec{x}+\vec{y},\ \vec{x}+\vec{z},\ \vec{y}+\vec{z}\}$ are linearly independent.

b) (6 pts) Show that $\{\vec{x}+\vec{y},\ \vec{x}+\vec{z},\ \vec{y}-\vec{z}\}$ are linearly dependent.

5. (12 pts) Find a basis for the kernel and the image of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ given by the matrix $A_T = \begin{pmatrix}1 & 0 & 1 & 1\\1 & 0 & 1 & 1\\1 & 1 & 3 & 0\end{pmatrix}$.

6. (12 pts) Enlarge the set $\{1+x, 1+x+x^3\}$ to a basis of $\mathcal{P}_3$. (Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree ≤ 3.)

1. Let V and W be vector spaces, let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a subset of V, and let $T : V \to W$ be a linear transformation.

a) (3 pts) Carefully define what it means to say that Span $S = V$.

b) (3 pts) Carefully define the set Image T.

c) (6 pts) Show that if Span $S = V$, then Image $T = \text{Span}\,\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$.

a) Span $S = V$ means

$\forall \vec{v} \in V$, $\exists\, c_1, \ldots, c_k \in \mathbb{R}$ such that $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$.

(Note technically, the above means $V \subset \text{Span}\, S$, but the reverse inclusion is trivial.)

b) $\text{Image}\,T = \{\vec{\omega} \in W \mid \exists\, \vec{v} \in V \text{ such that } \vec{\omega} = T(\vec{v})\}$.

c) Given: Span $S = V$

Let us show $\text{Image}\,T \overset{?}{\subset} \text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$

Let $\vec{\omega} \in \text{Image}\,T$. We want to show $\vec{\omega} \overset{?}{\in} \text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$.

Since $\vec{\omega} \in \text{Image}\,T$, $\exists\, \vec{v} \in V$ s.t. $\vec{\omega} = T(\vec{v})$.

This $\vec{v}$ can be expressed as $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ for some $c_1, \ldots, c_k \in \mathbb{R}$ (because $V = \text{Span}\, S$).

But then $\vec{\omega} = T(\vec{v}) = T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k)$

$= c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k)$ (T is linear)

So we have expressed $\vec{\omega}$ as a linear combination of $T(\vec{v}_1), \ldots, T(\vec{v}_k)$, so $\vec{\omega} \in \text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$, as desired.

Conversely, Let us show $\text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\} \overset{?}{\subset} \text{Image}\,T$.

Let $\vec{\omega} \in \text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$. We want to show, $\vec{\omega} \overset{?}{\in} \text{Image}\,T$.

We know $\exists\, c_1, \ldots, c_k \in \mathbb{R}$ s.t. $\vec{\omega} = c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k)$.

But then, by linearity of T, $\vec{\omega} = T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k)$.

This shows that $\vec{\omega}$ is the image under T of $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$.

Thus $\vec{\omega} \in \text{Image}\,T$, as desired.

The above two paragraphs show: $\text{Image}\,T = \text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$.

[Note the converse, $\text{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\} \overset{?}{\subset} \text{Image}\,T$, can be shown by arguing that $\forall i$, $T(\vec{v}_i) \in \text{Image}\,T$, and that Image T is a subspace. So Image T contains the span of the $T(\vec{v}_i)$'s, since the span is the smallest subspace containing the $T(\vec{v}_i)$'s.]

2. a) (8 pts) Show that the set $S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\4\\1\end{pmatrix}\}$ is a basis for $\mathbf{R}^3$.

b) (4 pts) Express the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ as a linear combination of the vectors in the above basis.

a) Let us show the vectors are linearly independent:

if $c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\1\\0\end{pmatrix} + c_3\begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$,

then
$$\begin{cases} c_1 + 2c_2 + 3c_3 = 0 & ① \\ c_2 + 4c_3 = 0 & ② \\ c_3 = 0 & ③ \end{cases}$$

Now ③ $\Rightarrow c_3 = 0$. combine with ②: this gives $c_2 = 0$. combine these 2 results with ①: this gives $c_1 = 0$.

Note if we have covered the theorem at this point, we can say: "these are 3 vectors in the 3-dimensional space $\mathbb{R}^3$; if they are linearly independent, they are automatically a basis for $\mathbb{R}^3$." However, at this point in the semester we had not yet seen this theorem, so we had to argue as follows:

Let us show that these vectors span $\mathbb{R}^3$. Given an arbitrary $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3$, we must show $\exists r_1, r_2, r_3 \in \mathbb{R}$ s.t. $r_1\begin{pmatrix}1\\0\\0\end{pmatrix} + r_2\begin{pmatrix}2\\1\\0\end{pmatrix} + r_3\begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

So we must solve
$$(*)\quad \begin{cases} r_1 + 2r_2 + 3r_3 = x_1 \\ r_2 + 4r_3 = x_2 \\ r_3 = x_3 \end{cases}$$
for r_1, r_2, r_3 (in terms of the "input" x_1, x_2, x_3).

It is easy to solve for r_3, then r_2, then r_1 (no need for full Gaussian elimination) to get:

$$r_3 = x_3$$
$$r_2 = x_2 - 4x_3$$
$$r_1 = x_1 - 2(x_2 - 4x_3) - 3x_3 = x_1 - 2x_2 + 5x_3.$$

This shows that the vectors span $\mathbb{R}^3$. Combining with their linear independence, we get that these vectors are a basis for $\mathbb{R}^3$.

b) we must solve $r_1\begin{pmatrix}1\\0\\0\end{pmatrix} + r_2\begin{pmatrix}2\\1\\0\end{pmatrix} + r_3\begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$. This is system (*) above with $x_1 = x_2 = x_3 = 1$. The solution is $\boxed{\begin{matrix} r_3 = 1 \\ r_2 = -3 \\ r_1 = 4 \end{matrix}}$. Thus $\boxed{\begin{pmatrix}1\\1\\1\end{pmatrix} = 4\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}2\\1\\0\end{pmatrix} + 1\begin{pmatrix}3\\4\\1\end{pmatrix}}$

3. Define the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_2 \\ x_1 \end{pmatrix}$.

a) (3 pts) Find the matrix A_T.
b) (4 pts) Show that T is injective but not surjective.
c) (5 pts) Find **one** linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^2$ such that $S \circ T = id_{\mathbf{R}^2}$ (there are many possible answers), and verify that $A_S A_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) $T(\vec{e_1}) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, and $T(\vec{e_2}) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}$. These are the columns of A_T. Thus $\boxed{A_T = \begin{pmatrix}1 & 2\\0 & 2\\1 & 0\end{pmatrix}}$.

b) T is injective: Let us show $\ker T \stackrel{?}{=} \{\vec{0}\}$. Of course $\{\vec{0}\} \subset \ker T$; we must show the converse.
If $\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \ker T$, then $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1+2x_2\\2x_2\\x_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, then $\left.\begin{matrix}x_1+2x_2=0\\2x_2=0\\x_1=0\end{matrix}\right\}$ these 2 show $x_1=x_2=0$.
thus $\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \ker T \Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. This shows $\ker T = \{\vec{0}\}$ as desired, & so T is injective

T is not surjective: it is enough to find ONE $\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} \notin$ Image T. There are many choices (anything with $y_1 \neq y_2 + y_3$ will do). The easiest example is:
Let us show that $\begin{pmatrix}1\\0\\0\end{pmatrix} \notin$ Image T. Indeed, if we had $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, for some $\begin{pmatrix}x_1\\x_2\end{pmatrix}$,
we would have $\begin{cases} x_1 + 2x_2 = 1 & (1) \\ 2x_2 = 0 & (2) \\ x_1 = 0 & (3) \end{cases}$. But then (2)+(3) says $x_1 + 2x_2 = 0$, which contradicts (1). There are no solns to this, so $\begin{pmatrix}1\\0\\0\end{pmatrix} \notin$ Image T, so T is not surjective.

c) There are many choices, we want $S(T\begin{pmatrix}x_1\\x_2\end{pmatrix}) = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ (since $id\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$)
so we want $S : \mathbb{R}^3 \to \mathbb{R}^2$ (linear), defined for any $S\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$, with the property that
$\forall x_1, x_2$, $S\begin{pmatrix}x_1+2x_2\\2x_2\\x_1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$. Some trial & error will show you that you can choose $\boxed{S\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}y_3\\ \frac{1}{2}y_2\end{pmatrix}}$ (since $S\begin{pmatrix}x_1+2x_2\\ \underbrace{2x_2}_{y_2}\\ \underbrace{x_1}_{y_3}\end{pmatrix} = \begin{pmatrix}x_1\\ \underbrace{x_2}_{\frac{1}{2}y_2}\end{pmatrix}$).
(Note: there are many other choices.)
Let us check: $A_S A_T \stackrel{?}{=} \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $A_S = \begin{pmatrix}0 & 0 & 1\\0 & \frac{1}{2} & 0\end{pmatrix}$ for our choice,
then $A_S A_T = \begin{pmatrix}0 & 0 & 1\\0 & \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 2\\1 & 0\end{pmatrix} = \begin{pmatrix}0\cdot1+0\cdot0+1\cdot1 & 0\cdot2+0\cdot2+1\cdot0\\ 0\cdot1+\frac{1}{2}\cdot0+0\cdot1 & 0\cdot2+\frac{1}{2}\cdot2+0\cdot0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

4. Given vectors $\vec{x}, \vec{y}, \vec{z} \in V$ with $\{\vec{x}, \vec{y}, \vec{z}\}$ linearly independent.
   a) (6 pts) Show that $\{\vec{x}+\vec{y},\ \vec{x}+\vec{z},\ \vec{y}+\vec{z}\}$ are linearly independent.
   b) (6 pts) Show that $\{\vec{x}+\vec{y},\ \vec{x}+\vec{z},\ \vec{y}-\vec{z}\}$ are linearly dependent.

a) Suppose $c_1(\vec{x}+\vec{y}) + c_2(\vec{x}+\vec{z}) + c_3(\vec{y}+\vec{z}) = \vec{0}$.
(We want to show $c_1 \overset{?}{=} c_2 \overset{?}{=} c_3 \overset{?}{=} 0$.)

Then $(c_1+c_2)\vec{x} + (c_1+c_3)\vec{y} + (c_2+c_3)\vec{z} = \vec{0}$,

thus $\begin{cases} c_1 + c_2 = 0 & ① \\ c_1 + c_3 = 0 & ② \\ c_2 + c_3 = 0 & ③ \end{cases}$ because $\{\vec{x}, \vec{y}, \vec{z}\}$ are linearly independent.

The system is equivalent to

$$\begin{array}{l} ① \\ ②-① \\ ③ \end{array} \begin{cases} c_1 + c_2 = 0 \\ -c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases} \underset{\text{exch}}{\iff} \begin{cases} c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \\ -c_2 + c_3 = 0 \end{cases}$$

$$\iff \begin{cases} c_1 - c_3 = 0 \\ c_2 + c_3 = 0 \\ 2c_3 = 0 \quad (\cdot\tfrac{1}{2}) \end{cases} \iff \begin{cases} c_1 - c_3 = 0 \\ c_2 + c_3 = 0 \\ c_3 = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$, as desired

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b) either try to solve a system analogous to part ⓐ, & look for a nontrivial solution, or notice by inspection that

$\vec{y} - \vec{z} = (\vec{x}+\vec{y}) - (\vec{x}+\vec{z})$. Thus one vector is a linear combination of the others, and they are linearly dependent.

If you prefer, you can write the dependency as:

$$\boxed{(-1)\cdot(\vec{x}+\vec{y}) + (1)\cdot(\vec{x}+\vec{z}) + (1)(\vec{y}-\vec{z}) = \vec{0}}$$

5. (12 pts) Find a basis for the kernel and the image of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ given by the matrix $A_T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 0 \end{pmatrix}$.

① $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \ker T \iff T\vec{x} = \vec{0} \iff \begin{cases} x_1 + x_3 + x_4 = 0 \\ \cancel{x_1 + x_3 + x_4 = 0} \quad \text{(redundant)} \\ x_1 + x_2 + 3x_3 = 0 \end{cases}$ (-1)

$\iff \begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \end{cases}$ here x_1, x_2 are bound, x_3, x_4 are free

$\iff \begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -2x_3 + x_4 \\ x_3 = x_3 \quad \text{(arbitrary)} \\ x_4 = x_4 \quad \text{(arbitrary)} \end{cases} \iff \vec{x} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$\iff \vec{x} \in \text{Span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. So $\boxed{\ker T = \text{Span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}}$

& this set is a basis since it is linearly independent; indeed, if $c_1 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ then $\begin{cases} -c_1 - c_2 = 0 \\ -2c_1 + c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases}$ so $c_1 = c_2 = 0$. So a basis for $\ker T$ is $\boxed{\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}}$.

② at this stage, we had not yet proved rank-nullity, so we must find a spanning set for $\text{Im}\, T$ that is linearly independent. As usual,

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \text{Im}\, T \iff \underbrace{\exists x_1, x_2, x_3, x_4 \in \mathbb{R}}_{\text{namely, } \exists \vec{x} \in \mathbb{R}^4}$ s.t. $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$\iff \exists x_1, \ldots, x_4 \in \mathbb{R}$ s.t. $\vec{y} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \iff \vec{y} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

But the 4 spanning vectors we have are not linearly independent! So choose a subset that will be a basis for $\text{Im}\, T$.

We note (by inspection or by direct computation) that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so we can remove $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & get $\text{Im}\, T = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Moreover, $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ so we can also remove $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ & get $\text{Im}\, T = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. These last 2 vectors are linearly independent: $c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_2 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$.

Hence $\boxed{\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ is a basis for } \text{Im}\, T}$.

<u>Note</u> there are many other choices of bases for each of $\ker T$, $\text{Im}\, T$.

6. (12 pts) Enlarge the set $\{1+x, 1+x+x^3\}$ to a basis of $\mathcal{P}_3$. (Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree ≤ 3.)

Using the technique we saw in class a few days before the quiz (there are of course other ways):

first, take the union with any basis of $\mathcal{P}_3$ ("after" our set, which is linearly indep to begin with), then remove any vector that is a linear combination of the previous vectors. Thus

$$\mathcal{P}_3 = \text{Span}\{1+x,\ 1+x+x^3,\ 1,\ x,\ x^2,\ \underline{\underline{x^3}}\}$$

note $x^3 = (1+x+x^3) - (1+x)$ so it can be removed

$$= \text{Span}\{1+x,\ 1+x+x^3,\ 1,\ \underline{\underline{x}},\ x^2\}$$

note $x = (1+x) - 1$, so it can be removed

$$= \text{Span}\{1+x,\ 1+x+x^3,\ 1,\ x^2\}$$

& finally check this set is linearly independent (not necessary if we use the theorem that 4 spanning vectors in a 4-dimensional space must be a basis):

if $c_1(1+x) + c_2(1+x+x^3) + c_3(1) + c_4(x^2) = 0$

then $c_2 x^3 + c_4 x^2 + (c_1+c_2)x + (c_1+c_2+c_3)1 = 0$

so
$$\begin{cases} c_2 = 0 \\ c_4 = 0 \\ c_1 + c_2 = 0 \\ c_1 + c_2 + c_3 = 0 \end{cases}$$

so $c_1 = 0$ also

so $c_3 = 0$ also

This shows linear independence. So a possible extension to a basis is $\boxed{\{1+x,\ 1+x+x^3,\ 1,\ x^2\}}$.

(Note that there are many other possibilities for the extension.)