Math 219 — Fall 2014
Final Exam, December 15. TIME: 2.5 hours

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	7	8	9	TOTAL/108

This exam booklet is also your answer sheet; **please answer question 0 on this page, and answer questions 1–9 inside the booklet.** The problems are repeated inside the booklet. There are extra blank sheets at the end for solutions. You can also use the back of any page for solutions or scratchwork. If you need to continue a problem beyond the first page where it appears, please INDICATE CLEARLY where the solution continues.

Read through the problems before starting, and decide which of them you wish to work on first. Do as much of the exam as you can, and budget your time wisely. Each of questions 1–9 counts for 12 points. The exam is closed book, and calculators are not allowed. You may bring to the exam one A4 sheet with handwritten notes or formulas on both sides.

Make sure to communicate your ideas by explaining all of your steps precisely and clearly.

Good luck!

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0. a) Your name: b) Your AUB ID#:

1. Define $T : \mathbf{R}^4 \to \mathbf{R}^3$ by the matrix $A_T = {}_{std}[T]_{std} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$.

a) Find a basis for each of $\ker T$ and $\operatorname{Image} T$.

b) Find new bases $\mathcal{B}$ for $\mathbf{R}^4$ and $\mathcal{C}$ for $\mathbf{R}^3$ such that ${}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

2. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & 2 & 0 & 3 \\ 3 & 4 & 3 & 10 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

a) Find the determinants $\det A$ and $\det B$.
b) What are the ranks $\operatorname{Rank} B$ and $\operatorname{Rank} A$ (in that order)? Justify.
c) (unrelated) Without doing any computations, why are A and B diagonalizable?

3. Let $W = \operatorname{Span}\{(1,1,0,0),(2,0,1,0),(2,0,0,1)\} \subset \mathbf{R}^4$. (We have written vectors as rows to save space.) Find the orthogonal projection $\operatorname{Proj}_W(30,0,0,0)$ of the vector $\vec{v} = (30,0,0,0)$. The numbers have been arranged so that the final answer does not involve any fractions.

4. Diagonalize the following two matrices, i.e., find the eigenvalues and a basis of eigenvectors: Do this if possible over $\mathbf{R}$, and if not possible over $\mathbf{C}$.

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

5. Let $T : V \to W$ be a linear transformation, and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V.

a) Show that T is surjective $\iff$ $\text{Span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\} = W$.

b) Show that T is injective $\iff$ $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ are linearly independent.

c) Show that T is bijective $\Rightarrow \dim W = \dim V$.

d) Give an example where $\dim W = \dim V$ but T is not bijective.

6. Recall that $\mathcal{P}_3$ is the space of polynomials $f(x)$ of degree at most 3. Define a linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ by

$$T(1) = 6, \qquad T(x) = 2x, \qquad T(x^2) = 4x + 2, \qquad T(x^3) = 12x^2 + 6x$$

and extending linearly. (This comes from the general formula $T(f) = (x+1)^2 f'' - 4xf' + 6f$, but you can ignore this fact.)

a) Find the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

b) Find a basis for each of $\text{Image}\, T$ and $\ker T$. Justify your reasoning. Make sure you give elements of $\mathcal{P}_3$.

c) Find the eigenvalues of T, and, for each eigenvalue, find one eigenvector. Again, these should be elements of $\mathcal{P}_3$.

d) Show that T is not diagonalizable.

7. Which of the following matrices are equivalent? Which are similar? Explain.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

8. Let V be a finite-dimensional inner product space over the field $\mathbf{F}$ (which can be $\mathbf{R}$ or $\mathbf{C}$), and let $T : V \to V$ be a self-adjoint linear transformation.

a) Assume $\mathbf{F} = \mathbf{C}$. Show that there exists a linear transformation $S : V \to V$ such that $S^2 = T$.

b) Assume $\mathbf{F} = \mathbf{R}$. Suppose that for all $\vec{v} \in V$, $\langle \vec{v}, T\vec{v} \rangle \geq 0$. Show that the conclusion of part (a) still holds about the existence of S.

9. Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be an **injective** linear transformation.

a) Show that there exists $S : W \to V$ such that $S \circ T = id_V$.

b) Show that if $\dim V = \dim W$, then $T \circ S = id_W$ as well. (This can be done even if you were not able to solve part (a); just accept part (a) and solve this part.)

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