**Math 219 — Fall 2014**
**Quiz II, October 30. TIME: 75 minutes**

**GRADES (each problem is worth 14 points):**

| 1 | 2 | 3 | 4 | 5 | 6 | TOTAL/84 |
|---|---|---|---|---|---|---|
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This exam booklet is also your answer sheet; **please answer question 0 on this page, and answer questions 1–6 inside the booklet.** The problems are repeated inside the booklet. There are extra blank sheets at the end for solutions. You can also use the back of any page for solutions or scratchwork. If you need to continue a problem beyond the first page where it appears, please INDICATE CLEARLY where the solution continues.

Read through the problems before starting, and decide which of them you wish to work on first. Do as much of the exam as you can, and budget your time wisely. Each of questions 1–6 counts for 14 points. The exam is closed book, and calculators are not allowed.

Make sure to communicate your ideas by explaining all of your steps precisely and clearly.

Good luck!

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**0.** a) Your name: b) Your AUB ID#:

**1.** Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ given by the matrix

$$A_T = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 0 \\ 4 & 8 & 5 & 4 \end{pmatrix}.$$

(7 pts) a) Find a basis for $\ker T$.

(7 pts) b) Show that $T$ is **not** surjective.

**2.** (7 pts) a) Show that $\left\{ \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis for $\mathbf{R}^3$.

(7 pts) b) Deduce in an easy way the **dimension only** $\dim \ker T$ of the kernel $\ker T$, where the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ is given by the matrix

$$A_T = \begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ 1 & 0 & 0 & 1 & 5 \\ 9 & 1 & 2 & 4 & 9 \end{pmatrix}.$$

Do **NOT** find a basis for $\ker T$!

**3.** Let $V$ be a vector space, and let $\vec{x}, \vec{y}, \vec{z} \in V$. Let $\vec{a} = \vec{x}$, $\vec{b} = \vec{y} + 2\vec{x}$, and $\vec{c} = \vec{z} - 4\vec{x}$.

(7 pts) a) Show that $\text{Span}\{\vec{a}, \vec{b}, \vec{c}\} = \text{Span}\{\vec{x}, \vec{y}, \vec{z}\}$.

(7 pts) b) Show that $\{\vec{x}, \vec{y}, \vec{z}\}$ is a basis for $V$ if and only if $\{\vec{a}, \vec{b}, \vec{c}\}$ is a basis for $V$.

**4.** (14 pts) Let $V$ be a vector space with $\dim V = n$. Let $W \subset V$ be a subspace with $\dim W = n-1$. Show that there exists a linear transformation $T : V \to \mathbf{R}$ such that $\ker T = W$.

**5.** Define a linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $T(f) = (x^2 - 2)f'' - 2xf' + 2f$.

(4 pts) a) Find $T(1)$, $T(x)$, and $T(x^2)$. (Check: $T(x^2) = -4$.)

(10 pts) b) Find a basis for each of $\text{Image}\, T$ and $\ker T$.

**6.** In $\mathbf{R}^3$, let $\vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\vec{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2\}$.

(4 pts) a) Show that $\vec{w}_1$ and $\vec{w}_2$ are orthogonal.

(6 pts) b) As usual, let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Find the projections $\text{Proj}_W \vec{e}_1$ and $\text{Proj}_{W^\perp} \vec{e}_1$.

(4 pts) c) Extend $\{\vec{w}_1, \vec{w}_2\}$ to an **orthogonal** basis $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ of $\mathbf{R}^3$. This means that you should find a possible vector for $\vec{w}_3$. Make sure to justify your reasoning.

1. Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ given by the matrix

$$A_T = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 0 \\ 4 & 8 & 5 & 4 \end{pmatrix}.$$

(7 pts) a) Find a basis for $\ker T$.
(7 pts) b) Show that $T$ is **not** surjective.

2. (7 pts) a) Show that $\left\{ \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis for $\mathbf{R}^3$.

(7 pts) b) Deduce in an easy way the **dimension only** $\dim \ker T$ of the kernel $\ker T$, where the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ is given by the matrix

$$A_T = \begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ 1 & 0 & 0 & 1 & 5 \\ 9 & 1 & 2 & 4 & 9 \end{pmatrix}.$$

Do **NOT** find a basis for $\ker T$! (Unless you are stuck and cannot find any other way.)

3. Let $V$ be a vector space, and let $\vec{x}, \vec{y}, \vec{z} \in V$. Let $\vec{a} = \vec{x}$, $\vec{b} = \vec{y} + 2\vec{x}$, and $\vec{c} = \vec{z} - 4\vec{x}$.
(7 pts) a) Show that $\text{Span}\{\vec{a}, \vec{b}, \vec{c}\} = \text{Span}\{\vec{x}, \vec{y}, \vec{z}\}$.
(7 pts) b) Show that $\{\vec{x}, \vec{y}, \vec{z}\}$ is a basis for $V$ if and only if $\{\vec{a}, \vec{b}, \vec{c}\}$ is a basis for $V$.

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