American University of Beirut
Final – Fall 2007-2008
Math 219

I – Let T: $R^3 \to R^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_2 \\ x_1 + 7x_3 \end{bmatrix}$.

a- Show that T is a linear transformation.
b- Write T as a matrix with respect to the standard basis of R^3.
c- Find a basis for the kernel and image of T.

II- Let U: $R^3 \to R^2$ and T: $R^3 \to R^3$ be the linear transformation given by the matrices representations: $U = \begin{pmatrix} -3 & 2 & -1 \\ -3 & -2 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & 0 & 3 \end{pmatrix}$.

a- Find the matrix of UoT.
b- Show that $\begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix} \in \text{Image}T \cap \text{Ker}U$

III- Let P_2 be the vector space of all polynomials of degree less or equal than two and $T : P_2 \to P_2$ be the linear transformation defined by:

$$T(p(x)) = T(a_0 + a_1 x + a_2 x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2$$

a- Find the eigenvalues of T with respect to the standard basis of P_2.
b- Find bases for the eigenspaces of T.

IV- Let u and v be in R^3. Define $<u, v> = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$.

a- Show that $<u, v> = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$ is an inner product
b- Use the Gram-Schmidt process to transform $u = (1,0,0)$, $v = (1,1,0)$ and $w = (1,1,1)$ into an orthonormal basis.

V- Define $u_1, u_2 \in R^2$ such that $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

a- Show that u_1, u_2 span R^2.

b- Let B = { e_1, e_2 } be the standard basis for R^2. Write e_1, e_2 as a linear combination

of u_1, u_2.

c- Let T: $R^2 \to R^3$ with T_B its matrix representation . And let

$T_B(u_1) = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$ and $T_B(u_2) = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$. Determine T_B.

VI- True or False. If it is True explain fully. If it is false give a counter-example.

a- If A is a 3x4 matrix then the system Ax = 0 has at least one independent variable.

b- T: $V \to W$ is a linear transformation . Let $v_1, v_2 \ldots v_n \in V$, then if $T(v_1), T(v_2) \ldots T(v_n)$ are linearly independent then $v_1, v_2 \ldots v_n$ are linearly independent.

c-

VII- Prove the following independent statements:

a- An nxn matrix is said to be idempotent if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix then $\lambda = 0$ or $\lambda = 1$.

b- Let P_3 be the vector space of all polynomials of degree less or equal than three. For any polynomial p(x) we define p'(x) to be the derivative of p(x). Let $g(x) = 1 + x - x^2 + 4x^3$. Show that C = {g(x), g'(x), g''(x), g'''(x)} is a basis for P_3.

c- Let A be a diagonalizable matrix whose eigenvalues are all 1 or -1. Show that A is invertible and that $A^2 = I$.

d- Let B be an n x n invertible matrix. Prove that

$$T : M_{n\times n} \to M_{n\times n}$$
$$A \to T(A) = AB$$

is an isomorphism.