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**Math 219, sections 3 and 4 — ~~Fall 2001-2002~~**
**Last year's final exam** (from Fall 2000-01)

1. a) Let $V$ be a vector space over $\mathbf{R}$. Define the notion of an inner product on $V$.
b) State and prove the Cauchy-Schwartz inequality (only for vector spaces over $\mathbf{R}$!).

2. Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be given by the matrix $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$.
a) Find a basis for each of $\ker T$ and $\operatorname{Image} T$.
b) Find the orthogonal projection of the vector $(3, 0, 0)$ onto $\operatorname{Image} T$.

3. Define $T : \mathbf{C}^2 \to \mathbf{C}^2$ by the matrix $\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$. Find a basis of $\mathbf{C}^2$ consisting of eigenvectors for $T$.



4. Define $T : \mathcal{P}_3 \to \mathbf{R}^4$ by $T(f) = \begin{pmatrix} f(1) \\ f'(1) \\ f(-1) \\ f'(-1) \end{pmatrix}$.
a) Find the matrix ${}_{\mathcal{A}}[T]_{\mathcal{B}}$, where $\mathcal{A}$ is the standard basis for $\mathbf{R}^4$ and $\mathcal{B} = \{1, x, x^2, x^3\}$.
b) Show that $T$ is an isomorphism. (You do not need to calculate $T^{-1}$.)

5. Let $V$ and $W$ be **finite-dimensional** vector spaces, and let $T : V \to W$ be a **surjective** linear transformation.
a) Show that there exists a linear transformation $U : W \to V$ such that $T \circ U = \mathrm{id}_W$. Hint: you can define $U$ on a basis for $W$ and use the linear extension theorem.
b) Give an example to show that $U \circ T$ need not equal $\mathrm{id}_V$.

6. Let $V$ be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation such that $T^2 = 0_V$ (Recall that $T^2 = T \circ T$, and that $0_V$ is the zero linear transformation).
a) Show that $\dim \operatorname{Image} T \le (1/2) \dim V$.
b) Give an example where $\dim \operatorname{Image} T \le (1/2) \dim V$, but $T^2 \ne 0_V$.

7. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by
$$Tf(x) = f(1-x), \qquad \text{e.g., } T(x^2 - 3x + 4) = (1-x)^2 - 3(1-x) + 4 = x^2 + x + 2.$$
a) Find the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ of $T$ with respect to the basis $\mathcal{B} = \{1, x, x^2\}$.
b) Show that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is similar to
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

8. Let $V$ be a finite-dimensional inner product space, and let $P : V \to V$ be a self-adjoint linear transformation such that $P^2 = P$.
a) Show that if $\lambda$ is an eigenvalue of $P$, then $\lambda = 0$ or $\lambda = 1$.
b) Show that $P$ is the orthogonal projection onto a certain subspace $W$ of $V$.

9. Let $V$ be a finite-dimensional **complex** inner product space, and let $T : V \to V$ be a unitary transformation. Recall that this means that $T^* = T^{-1}$, which is a fancy way of saying that $T$ is an isometry: for all $\vec{v}, \vec{w} \in V$, $\langle T(\vec{v}), T(\vec{w}) \rangle = \langle \vec{v}, \vec{w} \rangle$.
a) Show that if $\lambda$ is a (complex) eigenvalue of $T$, then $|\lambda| = 1$.
b) Show that $T$ is diagonalizable, by imitating the proof of the spectral theorem.