Professor K. Khuri-Makdisi

Math 219, sections 3 and 4 — Fall 2001–2002
Final Exam, January 31, 2002

Instructions: Please **write your name and AUB ID number on this page (question 0)**, and answer questions 1–10 inside the booklet. You may continue your solutions on the backs of pages and on the extra blank sheets at the end. In that case, please indicate where you are continuing your work. If you must hand in extra sheets, please **write your name on each extra sheet** that you hand in.

This is a long exam; do as much of it as you can, and budget your time wisely. Each of questions 1–10 counts for 10 points. You must **justify your steps** in order to receive full credit. **None of the problems requires much computation.**

Note: Except in problem 5, the field of scalars is $\mathbf{F} = \mathbf{R}$.

GOOD LUCK!

0. a) Your name: b) Your AUB ID#:

1. a) Carefully define what it means for two matrices M and M' to be similar.
b) Show that if M and M' are similar matrices, then they have the same determinant.
c) Give an example of two matrices with the same determinant but different ranks (so they will definitely not be similar!).

2. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be given by the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. Find an **orthonormal** basis for each of $\ker T$ and $\operatorname{Image} T$.

3. Let $\mathbf{B_1}, \ldots, \mathbf{B_k}$ be an orthogonal set of nonzero vectors in the inner product space V. Show that $\{\mathbf{B_1}, \ldots, \mathbf{B_k}\}$ are linearly independent. (Don't just quote the theorem from class; prove it.)

4. Define the linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ by $T(f) = xf' - f''$.
a) Find the matrices ${}_{\mathcal{A}}[T]_{\mathcal{A}}$ and ${}_{\mathcal{B}}[T]_{\mathcal{B}}$, where $\mathcal{A} = \{1, x, x^2, x^3\}$ and $\mathcal{B} = \{1, x+1, x^2, x^3+x\}$.
b) Show that T is diagonalizable. (It is possible to do this without finding an explicit basis of eigenvectors.)

5. Define $T : \mathbf{C}^3 \to \mathbf{C}^3$ by the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & -4 \\ 1 & 1 & 1 \end{pmatrix}$. Find a basis of $\mathbf{C}^3$ consisting of eigenvectors of T.

6. a) Show that the matrix $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \\ 0 & 2 & 2 \end{pmatrix}$ is invertible. (You do not need to explicitly calculate the inverse matrix.)
b) Find the **dimension only** of the kernel of the matrix below:

$$\dim\ker \begin{pmatrix} 1 & 3 & 10 & 2 & 100 \\ 2 & 6 & 20 & 5 & 400 \\ 0 & 2 & 30 & 2 & 200 \end{pmatrix}.$$

7. Let V be an n-dimensional inner product space, and let $W \subset V$ be a k-dimensional subspace. Find the characteristic polynomial of P, where we define $P : V \to V$ to be the orthogonal projection onto W:

$$P(\mathbf{A}) = \operatorname{Proj}_W(\mathbf{A}).$$

8. Recall that $\mathbf{F} = \mathbf{R}$. Let $\{\mathbf{U_1}, \mathbf{U_2}, \ldots, \mathbf{U_n}\}$ be an **orthonormal** basis for V. Show that for all $\mathbf{A} \in V$,

$$\|\mathbf{A}\|^2 = \langle \mathbf{A}, \mathbf{U_1} \rangle^2 + \langle \mathbf{A}, \mathbf{U_2} \rangle^2 + \ldots + \langle \mathbf{A}, \mathbf{U_n} \rangle^2.$$

(For 3 bonus points: state and prove the analogous result in case $\mathbf{F} = \mathbf{C}$. Caution: over $\mathbf{C}$, the correct statement is slightly different from the above equation.)

9. a) Let M be the matrix $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$. Show that M is diagonalizable.

b) Show that M is similar to a diagonal matrix of the form $\begin{pmatrix} \lambda & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$, where $\lambda \neq 0$. (Hint: what is the dimension of $\ker M$?)

c) Find the eigenvector corresponding to λ, and determine the value of λ.

10. Let $g \in \mathcal{P}_n$ be a polynomial (of degree at most n). Prove that there exists a unique polynomial $f \in \mathcal{P}_n$ (again of degree at most n), such that $f' + f = g$.