Math 219 — Fall 2004
Final Exam, February 1, 2005. Time: 2 1/2 hours

Remarks: For the final, please answer the problems in the examination booklets provided. You may work on the problems in any order, but please make it clear which problem you are solving on any given page.

Each problem below is worth **15 (fifteen)** points, for a total of 135 points (and 10 bonus points). The problems are arranged **roughly** in order of increasing difficulty. Make sure to budget your time wisely.

Good luck!

1. Define $T : \mathbf{R}^4 \to \mathbf{R}^3$ by the matrix ${}_{std}[T]_{std} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$.

   (i) Find a basis for $\ker T$.
   (ii) Find an **orthonormal** basis for $\operatorname{Image} T$.

2. (i) Compute the following determinant: $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 \\ 1 & 2 & 6 & 7 \end{pmatrix}$.

   (ii) Use the result of part (i) to show that the following vectors in $\mathbf{R}^7$ are linearly independent (this can be done without any further calculation, but if you're stuck you can always calculate directly):

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 5 \\ 6 \\ 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 7 \\ 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Define $T : \mathbf{C}^2 \to \mathbf{C}^2$ by the matrix $M = {}_{std}[T]_{std} = \begin{pmatrix} 2 & -1+2i \\ 1 & 2i \end{pmatrix}$.

   (i) Find the eigenvalues of T, and for each eigenvalue, a corresponding eigenvector.
   (ii) Find an invertible matrix P such that $P^{-1}MP$ is diagonal. (You do not need to find P^{-1}.)

4. Define two linear transformations $T, U : \mathcal{P}_2 \to \mathcal{P}_2$ by

$$T(f) = f(2+x), \qquad U(f) = f(2-x).$$

(For example, $T(x^2+3x+1) = (2+x)^2 + 3(2+x) + 1 = x^2 + 7x + 11$.) Use the basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathcal{P}_2$.

   (i) Find ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[U]_{\mathcal{B}}$.
   (ii) Show that T is not diagonalizable.
   (iii) Show that U is diagonalizable, and find a basis of $\mathcal{P}_2$ consisting of eigenvectors of U.

5. Let M be an $n \times n$ matrix, which we also view as a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. We write $P_M(\lambda)$ for the characteristic polynomial of M.

   (i) Show that μ is a root of P_M if and only if μ is an eigenvalue of M.
   (ii) [independent of part (i)] Show that if M and N are similar matrices, then they have the same characteristic polynomial.

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6. Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ and $U : \mathbf{R}^4 \to \mathbf{R}^2$ be linear transformations whose composition is the zero linear transformation: i.e., $UT = \mathbf{0}$.

(i) Show that if U is surjective, then T cannot be injective.

(ii) Give a specific example of $T : \mathbf{R}^3 \to \mathbf{R}^4$ and $U : \mathbf{R}^4 \to \mathbf{R}^2$ with $UT = \mathbf{0}$ as above, for which T is injective and U is nonzero (but of course, U cannot be surjective).

7. Let V be an inner product space, and let $\vec{0} \neq v \in V$. Define a linear transformation $T : V \to V$ by

$$T(x) = x - 2\frac{\langle x, v\rangle}{\langle v, v\rangle}v.$$

(i) Show that T is the reflection about the hyperplane $\{v\}^{\perp}$. [Suggestion: find $T(v)$, and find $T(z)$ when $z \perp v$.]

(ii) Show that T is an isometry: for all x, we have $\|T(x)\| = \|x\|$.

8. Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. We also view M as a linear transformation from $\mathbf{R}^6$ to $\mathbf{R}^6$.

(i) Find the rank of M.

(ii) Show that M is diagonalizable.

(iii) Show that M is similar to a diagonal matrix of the form $\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$, with λ_1, λ_2 nonzero.

(iv) Bonus (5pts): Find λ_1 and λ_2. [One possible way to avoid doing long calculations: work inside a suitable two-dimensional subspace $W \subset \mathbf{R}^6$ which is invariant for M. There are other ways that avoid doing long calculations.]

9. Let V be a finite-dimensional inner product space, and assume given a **self-adjoint** linear transformation $T : V \to V$ such that $T^3 = T$.

(i) Show that the only possible eigenvalues of T are $\lambda = 0$, 1, or -1. Write E_0, E_1, E_{-1} for the corresponding eigenspaces.

(ii) Show that every vector $v \in V$ has a decomposition $v = v_0 + v_1 + v_{-1}$, with $v_0 \in E_0$, $v_1 \in E_1$, and $v_{-1} \in E_{-1}$.

(iii) Define a linear transformation $P : V \to V$ by $P = (1/2)(T^2 + T)$. Show that P is the orthogonal projection onto E_1.

(iv) Bonus (5pts): Find another "polynomial" $Q = aT^2 + bT + cI$ for suitable $a, b, c \in \mathbf{R}$, such that Q is the orthogonal projection onto E_0.