AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department
Math 219 - Final
Fall 2005-2006

Name:................................ **ID:**...............................

Instructor: Ms. Diana Audi **Time: 2 hrs**

I- (**20 points**) Let $V = C[0,1]$ be the vector space of all continuous functions on $[0,1]$. We define an inner product $< .,. >$ on V to be:
$< f, g >= \int_0^1 f(x)g(x)dx, \quad for all f, g \in V.$

- a- Let $W = span\{1, x\}, \quad h = 4 + x^2$, find the orthogonal projection of h on W.
- b- Find the distance $d(h, W)$ from h to W.
- c- Find an upper bound on $\int_0^1 x^{100} \sqrt{x^2+1}dx$.

II- (**35 points**)Let $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ be the mapping defined by $T(A) = A^T$, the transpose of A, and F is the field of complex numbers.

- a- Verify that T is a linear operator on $M_{2\times 2}(F)$.
- b- Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the standard basis for $M_{2\times 2}(F)$,find $[T]_\beta$.
- c- Find the eigen values of T and describe the corresponding eigen spaces.
- d- Show that T is digonalizable, and find a basis β' of $M_{2\times 2}(F)$ consisting of eigen vectors of T.
- e- Find an invertible matrix P such that $[T]_{\beta'} = P^{-1}[T]_\beta P$.

III- (**25 points**)Let $A_{n\times n}$ matrix ,with $a_{ij} = 1, \quad for\ all \quad i, j$.

$$A = \begin{pmatrix} 1 & 1...... & 1 \\ 1 & 1...... & 1 \\ 1 & 1...... & 1 \\ . & .... & . \\ . & .... & . \\ 1 & 1....... & 1 \end{pmatrix}$$

- a- Find $rank(A)$ and $nullity(A)$.
- b- Deduce that $\lambda_1 = 0$ is an eigen value of A,and find the **dimension** of eigen space corresponding for $\lambda_1 = 0$.
- c- Show that $\lambda_2 = n$ is an eigen value of A,and find the **dimension** of eigen space corresponding for $\lambda_2 = n$.(Hint: find $A \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$)

d- Show that A is diagonalizble .

IV- **(25 points)** Let $T : F^3 \to F^3$ be a linear operator on F^3, where F is the field of complex numbers. Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis of F^3.
Let $[T]_\beta = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -2 \end{pmatrix}$

a- Show that T is a self adjoint linear operator.

b- Deduce that T is diagonalizable.

c- Find the eigen values of T.

d- let $||T|| = \max_{||x=1||} ||T(x)||$, Find $||T||$?

V- **(25 points)**Let V be an inner product space over $\mathbf{R}$ .Let $T : V \to V$ be a linear operator on V.

a- Show that $< x, y >= \frac{1}{4}(||x + y||^2 - ||x - y||^2)$.

b- Using part(a), show that $||T(x)|| = ||x||\ for\ all\ x \in V$ **if and only if** $< T(x), T(y) >=< x, y > \quad for \quad all\ x, y \in V$.

c- If T is an isometry i.e $(||T(x)|| = ||x||\ for\ all\ x \in V)$, find the eigen values of T.

d- If T is an isometry ,show that T is one to one.

VI- **(15 points)** 1- Let $T : V \to W$ be a linear transformation, with $dim(V) = n$ Let $\{v_1, ..v_k\}$ be a basis for $kerT$, and $\{v_1, ...v_k, v_{k+1}, ..v_n\}$ a basis for V.

a- Show that $\{T(v_{k+1}), .., T(v_n)\}$ is a basis for the range of T.

b- Deduce that $Rank(T) + N(T) = n$

2-**(15 points)**- Let $T : V \to V$ be a linear **operator**, with $dim(V) = n$ and $rank(T^2 = ToT) = rank(T)$.

a- show that $N(T^2) = N(T)$.

b- Deduce that $KerT^2 = KerT$.

c- Show that $V = KerT \oplus Range(T)$.

Application:(25 points) Let V be an inner product space, and W be a subspace of V. Let $T : V \to V$ be a linear operator on V, with $T(v) = proj_W(v)\ for\ all\ v \in V$.

a- Find $range(T)$ and $Ker(T)$.

d- Find $T^2 = ToT$.

c- Show that $||T(v)|| \leq ||v||\ for\ all\ v \in V$.

VII- **(15 points)**Let $T : \mathbf{R^3} \to \mathbf{R^3}$ be a linear operator on $\mathbf{R^3}$, with $T^3 = ToToT = 0$. Let $a \in \mathbf{R^3}$ be such that $T^2(a) \neq 0$.
Show that $\{a, T(a), T^2(a)\}$ is a basis for $\mathbf{R^3}$.

VIII- (**Bonus 10 points**)Let T, S be linear **operators** on a **finite** dimensional vector space V.

a- If ToS is one to one, prove that S is one to one.

b- If ToS is onto ,prove that T is onto.

c- Deduce that if ToS is invertible ,then T and S are invertible.