



AMERICAN UNIVERSITY OF BEIRUT

Spring 2000-01
Time : 2 Hours.
Prof. H. Abu-Khuzam

MATHEMATICS 219
FINAL EXAMINATION

NAME -----
ID# -----
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1. Let $T: P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ be the linear transformation defined by

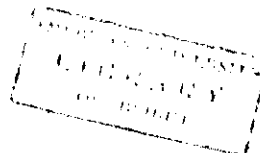
$$T(p(x)) = x^2 p'(x) + 3p(x)$$

a. Find the matrix of T relative to the standard ordered bases of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$.

[6 points]

b. Find a basis for $\text{Ker } T$. Find $\dim(\text{Im } T)$.

[6 points]



2. Let $A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

a. Find the eigenvalues and a basis for each eigenspace of A.

[8 points]

b. Deduce that there is a basis of \mathbb{R}^3 consisting of eigenvectors of A and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP=D$. (Do not verify)

[4 points]

2c. Use part (b) above to find an orthonormal basis of \mathbb{R}^3 consisting of eigenvectors of A.
[3 points]

3. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A must be invertible. (Hint: Use Determinants)

[7 points]

4. Show that an orthogonal set of nonzero vectors in an inner product space V is linearly independent.

[8 points]

5. Let $T: V \rightarrow V$ be a linear operator on a finite dimensional vector space V . Show that
 $(\text{Ker } T) \cap (\text{Im } T) = \{0\} \Rightarrow V = (\text{Ker } T) + (\text{Im } T)$

[7 points]

6. Use the row echelon form of the augmented matrix of the following system

$$\begin{aligned}x + 2y + z &= 3 \\x + 3y - z &= 4 \\x + 2y + (a^2 - 8)z &= a.\end{aligned}$$

to find the value(s) of a for which the system has

- a. no solution
- b. a unique solution
- c. infinitely many solutions.

[10 points]

7. Let $T: V \rightarrow W$ be a linear transformation. Show that If A_1, A_2, \dots, A_n are vectors in V , such that $T(A_1), T(A_2), \dots, T(A_n)$ are linearly independent in W , then A_1, A_2, \dots, A_n are linearly independent in V .

[8 points]

8. Let V be a vector space of dimension 3, and let U be a subspace of V of dimension 2. Show that there exists a linear transformation $T: V \rightarrow V$ such that $T(A) = 0$ for all $A \in U$, and T is not the zero map on V .

[6 points].

9. Answer **TRUE** or **FALSE** only: [3 points for each correct answer , 0 for no answer, and -1 for each wrong answer].

- a. ----- Let A be a 4×4 matrix such that $(\text{rank } A)=3$, then $|A| = 0$
- b. ----- The subspace of \mathbf{R}^3 spanned by the vectors $(1,0,-1)$, $(2,1,-3)$, and $(5,2,-7)$ has dimension 3.
- c. ----- Similar matrices have the same determinant.
- d. ----- Any set of 2 vectors in \mathbf{R}^3 can be extended to a basis of \mathbf{R}^3 .
- e. ----- If N is a square matrix such that $N^2=0$, then $I-N$ is invertible
- f. ----- The space of all symmetric 3×3 matrices has dimension 5.
- g. ----- If A , B , and C are $n \times n$ matrices such that $AB=AC$, then $B=C$.
- h. ----- Let W be a subspace of a vector space V , then the set $U = \{A \in V : A \notin W\}$ is a subspace V .
- i. ----- Let A be an $n \times n$ matrix such that $AA^T=I$, where I is the identity $n \times n$ matrix, then $|A| = 1$.

[27 points]