Prof. N. Haidar — FACULTY OF ARTS & SCIENCES , A. U. B.
Sec. **1** — **Math 219** — June 20, 2001
(2-nd Semester , 2000-2001)
Time : 2 hours — **FINAL EXAM** — (closed book)

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❺I. Suppose that $\mathbf{V}$ is a finite-dimensional vector space and $S \subsetneq \mathbf{V}$. Apply the basis extension theorem to prove that $\exists$ a subspace $\mathcal{T}$ of $\mathbf{V}$ such that $S \cap \mathcal{T} = \{\mathbf{0}\}$ and $S + \mathcal{T} = \mathbf{V}$.

❹II. Let $T : \mathbf{V} \to W$ be a linear transformation. Show that if $A_1, A_2, \ldots, A_n \in \mathbf{V}$ such that $T(\bar{A}_1), T(\bar{A}_2), \ldots, T(\bar{A}_n)$ are LID, then $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ are LID.

❹III. Let T be a linear transformation on a finite-dimensional vector space $\mathbf{V}$. If $ker\mathrm{T} \cap image\mathrm{T} = \{\mathbf{0}\}$, show that $\mathbf{V} = ker\mathrm{T} + image\mathrm{T}$.

❹IV. Given $V = (\mathbf{V}, <,>)$, with $\mathbf{V}$ a finite-dimensional vector space. Prove that an orthogonal set of nonzero vectors in V is L I D.

❻V. Let $T : R^2 \to R^2$ be such that $T(x,y) = (y-x , y+x)$, $\forall (x,y) \in R^2$.

(i) Show that T is an isomorphism.

(ii) Find a formula for T^{-1} to prove that $2^{-n/2}\, T^n$ is involutory , i.e. $2^{-n}T^{2n} = \mathbf{I}$, $\forall n \in Z = \{0,1,2,3, \ldots, \infty\}$.

❼VI. (i) Determine the eigenspace associated with the smallest eigenvalue of A when

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}; \quad \text{(ii) prove that} \quad \begin{vmatrix} a & -b & 0 & . & 0 & 0 \\ 0 & a & -b & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & -b & 0 \\ 0 & 0 & 0 & . & a & -b \\ -b & 0 & 0 & . & 0 & a \end{vmatrix} = a^n - b^n .$$

❻VII. Prove that if $V = (\mathbf{V}, <,>)$, with $\mathbf{V}$ a finite-dimensional vector space and if $W \subsetneq V$, then $W \oplus W^{\perp} = V$.

❽VIII. Consider the vector $\bar{S} = (6,0,12) \in R^3$. Let W be the subspace of R^3 spanned by $\bar{A}_1 = (1,0,1)$ and $\bar{A}_2 = (2,1,0)$.

(i) Decompose $\bar{S}$ into the sum of a vector $\bar{B} \in W$ and a vector $\bar{C} \in W^{\perp}$.

(ii) Find the projection matrix for W and use it to find the projection of $\bar{S}$ on W.

(iii) Apply the Gram-Schmidt process to construct an orthonormal basis for W.

❻IX. Given the linear algebraic system of equations

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 2 & (\beta^2 - 8) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ \beta \end{pmatrix}.$$

(i) Determine the value of β for which the system becomes inconsistent.

(ii) Find the minimal norm solution (the pseudo solution) of the previous inconsistent system.
