Mathematics 219 Final Exam. (August 16, 2004) **N. Nahlus**
Time: 130 minutes

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1. Let A and B be two matrices with 7 columns such that rank (A)=2, rank (B)=3, rank $\begin{pmatrix} A \\ - \\ B \end{pmatrix}$=5.

Let A^*, B^*, C^* be the *null spaces* of A, B and $\begin{pmatrix} A \\ - \\ B \end{pmatrix}$ respectively. (i) *Use the rank-nullity theorem* to find the dimensions of A^*, B^*, and C^*. (ii) *Deduce* that $A^* + B^* = R^7$.

2. If {a, b, c} is a basis of a vector space V, show that {a+c, b+c, c} is also a basis of V.

3. Let $f: V \to W$ be linear transformation of vector spaces such that $\{f(v_1), \ldots\ldots, f(v_n)\}$ are linearly independent. Show that $\{v_1, \ldots\ldots, v_n\}$ are linearly independent vectors in V.

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4. Prove the following theorem ***Without*** *using the rank-nullity Theorem.*
Theorem: Let $f: V \to W$ be linear transformation of vector spaces. If f is 1-1 and dimV=dimW, then f is an isomorphism. (**Reminder**: You may use any result proven in class unlessyou are asked to prove it)

5. Find a 5×5 matrix A such that rank A=3 and rank $A^2 = 2$.
(Hint: Apply the Linear Extension Theorem)

6. Show that similar matrices have the same determinant, the same trace, and the same eigen values.

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7. If 4 is an eigen value of T^2, show that 2 *or* -2 is an eigenvalue of T. (Hint: Use upper triangulization)

8. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **(i)** Find the eigen values of A and a basis for each eigen space of A.

(ii) Show that A is diagonalizable and find the exact relation between A , P and D. (Do **not** calculate P^{-1}).

9. (a) What do we know about arbitrary symmetric matrices regarding eigenvalues & diagonalization ?
(b What do we know about the minimal polynomial with respect to diagonalization?
(c) Apply the Cauchy-Schwarz inequality the inner product space of continous functions on [a, b] (with the well-known way of "dotting" functions), then find an upper bound on $\int_0^1 x^{500} e^{2x} dx$.

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10. True-False. **If false, give a counter example**

(a) If a set of 3 vectors (say in R^3) have the property that each 2 of them are linearly independent, then this set is linearly independent.
(b) For an $m \times n$ matrix A, A & $RRE(A)$ have the same null space & row space.
(c) For an $m \times n$ matrix A, A & $RRE(A)$ have the same column space.
(d) A 4×4 matrix with 4 distinct eigenvalues is diagonlizable.

11. Suppose every vector in V is an eigen vector for T where T is a linear operator on a vector space V, show that T is a scaler multiple of the identity I. (Hint: Use a, b, a+b)

12. Let V=A+B=A+C where A and B are subspaces of a vector space V.
Suppose dimA=∞, dimB=5, and $A \cap B = 0 = A \cap C$. Show that dimC=5.