CMPS 256: Advanced Algorithms and Data Structures

Chapter 3
Exercises & Solutions

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Chiraz BenAbdelkader

Exercise 1

Compare the rate of growth of each pair of functions f and g below.

f	**g**	
$5n^2 + 100n$	$3n^2 + 2$	$f = ?(g)$
$\log_3(n^2)$	$lg(n^3)$	$f = ?(g)$
n^{lg4}	$3^{lg\ n}$	$f = ?(g)$
lg^2n	$n^{1/2}$	$f = ?(g)$

Solution

f	**g**	$\lim_{n \to \infty} f(n)/g(n)$	Answer
$5n^2 + 100n$	$3n^2 + 2$	$5/3$	$f = \Theta(g)$
$\log_3(n^2)$	$\lg(n^3)$	$2/3 * \lg 3$	$f = \Theta(g)$
$n^{\lg 4}$	$3^{\lg n}$	∞	$f = \omega(g)$
$\lg^2 n$	$n^{1/2}$	0	$f = o(g)$

Exercise 2

True or false?

$$\mathbf{f(n)=O(g(n)) \Rightarrow f(n)+g(n)=\Omega(f(n))}$$

If true, give a proof based on formal definitions of order notation.

If false, give a counter example.

<u>Solution</u>: True. There are many ways to prove this.

Exercise 2 (Cont.)

First proof method:

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n)) \quad \text{(transpose symmetry)}$$

Also, $f(n) = \Omega(f(n))$

$$\therefore f(n) + g(n) = \Omega(f(n)) \qquad (\Omega(f(n)) \text{ is closed under addition})$$

Second proof method:

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n)) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} > 0 \Rightarrow \left[\lim_{n\to\infty} \frac{g(n)}{f(n)}\right] + 1 > 0$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{g(n)}{f(n)} + 1\right] > 0 \Rightarrow \lim_{n\to\infty} \frac{g(n) + f(n)}{f(n)} > 0 \Leftrightarrow g(n) + f(n) = \Omega(f(n))$$

Exercise 2 (Cont.)

Third proof method:

$$
\begin{aligned}
f(n) = O(g(n)) &\Leftrightarrow \exists c, n_0 > 0 \mid f(n) \leq cg(n), \forall n \geq n_0 \\
&\Leftrightarrow \exists c, n_0 > 0 \mid g(n) \geq (\tfrac{1}{c}) f(n), \forall n \geq n_0 \\
&\Leftrightarrow \exists c, n_0 > 0 \mid g(n) + f(n) \geq (\tfrac{1}{c} + 1) f(n), \forall n \geq n_0 \\
&\Leftrightarrow \exists c', n_0 > 0 \mid g(n) + f(n) \geq c' f(n), \forall n \geq n_0 \\
&\Leftrightarrow g(n) + f(n) = \Omega(f(n))
\end{aligned}
$$

Exercise 3

Order these functions asymptotically from slowest to fastest rate of growth:

1. $\sqrt{n}\,\lg^2 n$
2. $20\lg(5n^4)$
3. $n^{2.999}$
4. $5^{1/\lg n}$
5. $\lg(n!)/\lg n$
6. $n+\sqrt[4]{n}$
7. $3\lg^2 n + n^{0.6}$
8. $n^3/\lg n$
9. $(n-100)!$
10. $n^{\lg\lg n}$

Exercise 3 (Cont.)

Solution Strategy:

- In any question of this form, first simplify each function as much as possible
  - Typically, simplify to well-known functions such as polynomial, poly-log, exponential, or factorial
- Use following rules from formula sheet to sort functions
  - Polynomials are easy to compare: for any a,b>0, if a>b then $n^a=\omega(n^b)$
  - So are exponentials: for any a,b>1, if a>b then $a^n=\omega(b^n)$
  - Also we know that: for any c>1, a>0, b>0

$$\mathbf{n^n = w(n!) = w(c^n) = w(n^b) = w(\log^a n)}$$

- If cannot use any of these rules, compare functions by computing the limit of their ratio

Exercise 3 (Cont.)

First step: simplify functions whenever possible

1. $\sqrt{n}\lg^2 n = \Theta(n^{0.5}\lg^2 n)$
2. $20\lg(5n^4) = 20\lg 5 + 80\lg n = \Theta(\lg n)$
3. $n^{2.999} = \Theta(n^{2.999})$
4. $5^{1/\lg n} = \Theta(1)$ (because this is a decreasing function!)
5. $\lg(n!)/\lg n = \Theta(n\lg n/\lg n) = \Theta(n)$ (because $f(n) = \Theta(g(n)) \Rightarrow \frac{f(n)}{h(n)} = \Theta(\frac{g(n)}{h(n)})$)
6. $n + \sqrt[4]{n} = \Theta(n)$
7. $3\lg^2 n + n^{0.6} = \Theta(n^{0.6})$
8. $n^3/\lg n = \Theta(n^3/\lg n)$
9. $(n-100)! = \Theta((n-100)!)$
10. $n^{\lg\lg n} = \Theta((\lg n)^{\lg n})$ (based on the formula $a^{\log_b n} = n^{\log_b a}$)

Exercise 3 (Cont.)

<u>Second step</u>: separate the functions into homogeneous groups in increasing order of growth

$$\text{Constants} : \Theta(1)$$

$$\text{Logarithms} : \Theta(\lg n)$$

$$\text{Poly-logs} : \text{None}$$

$$\text{Polynomials} : \Theta(n) \;\; \Theta(n^{0.6}) \;\; \Theta(n^{2.999})$$

$$\text{Exponentials} : \text{None}$$

$$\text{Factorials} : \Theta(n-100)!$$

$$\text{Other} : \Theta(n^{0.5} \lg^2 n) \;\; \Theta(n^3 / \lg n) \;\; \Theta\left(\lg n\right)^{\lg n}$$

Exercise 3 (Cont.)

<u>Third step</u>: sort the functions in each group

Use theorems (in formula sheet); otherwise compute the limit of ratio of pairs of functions

For example, we have the following:

$$\lim_{n\to\infty} \frac{n^3/\lg n}{n^{0.5}\lg^2 n} = \lim_{n\to\infty} \frac{n^{2.5}}{\lg^3 n} = \infty \Rightarrow \boxed{n^3/\lg n = w(n^{0.5}\lg^2 n)}$$

$$\lim_{n\to\infty} \frac{\lg\left[n^3/\lg n\right]}{\lg\left[(\lg n)^{\lg n}\right]} = \lim_{n\to\infty} \frac{3\lg n - \lg\lg n}{\lg n \lg\lg n} = \lim_{n\to\infty}\left[\frac{3}{\lg\lg n} - \frac{1}{\lg n}\right] = 0$$

$$\Rightarrow \lg\left[(\lg n)^{\lg n}\right] = w(\lg\left[n^3/\lg n\right]) \Rightarrow \boxed{(\lg n)^{\lg n} = w(n^3/\lg n)}$$

Exercise 4

Given two functions f(n) and g(n), let

$$h(n) = \max(f(n),g(n))$$

Prove that: **h(n) = Q(f(n)+g(n))**

Solution:

Like in Exercise 2, you should use formal definitions of order notation in your proof.

Hint: use this definition of Θ-notation

$$h(n)=\Theta(f(n)+g(n)) \iff$$

$$\exists c_1,c_2,n_0>0 \mid c_1(g(n)+f(n)) \leq h(n) \leq c_2(f(n)+g(n))$$

(to be finished as homework)