Math 201

Quiz # 1, 22 February, 2016

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Answer all questions.Only one examination booklet is supplied.Use a different page for each question.Use pages marked "rough work" if necessary.Maximum time allowed is 70 minutes. No Calculators allowed.

1. (24 pts.) Find each of the following limits:

$$(a) \lim_{n\to\infty}\left(\frac{3n-2}{3n+2}\right)^n, \ (b) \lim_{n\to\infty}(-1)^n\frac{\cos(\ln n)}{\sqrt{n}}$$

$$(c) \lim_{n\to\infty}\frac{1^{-\frac{1}{4}}+2^{-\frac{1}{4}}+3^{-\frac{1}{4}}+\cdots+n^{-\frac{1}{4}}}{n^{\frac{3}{4}}}.$$

2. (24 pts.) Decide the convergence or divergence of each of the following series. In each case give the name of the test you used..

$$(a) \sum_{n=1}^{\infty}\frac{2+\sin n}{n\sqrt{n}}, \ (b) \sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^{2/3}, \ (c) \sum_{n=1}^{\infty}\left(e^{\frac{\ln n}{n\sqrt{n}}}-1\right).$$

3. (12 pts.) Let S_n be the n^{th} partial sum of the series of positive terms $\sum_{n=1}^{\infty}a_n$, and suppose that $\lim_{n\to\infty}a_ne^{S_n}=1$. Find $\lim_{n\to\infty}a_n$ and prove your answer.

4. (16 pts.) Consider the series

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}\left(\frac{x+3}{2}\right)^n.$$

Find all values of x for which the series is convergent, distinguishing between absolute and conditional convergence.

5. (24 pts.) Let $f(x)=\frac{3}{4+3x}$. (a) Find the Taylor series expansion of f about the point $a=1$;(b) then find $f^{(n)}(1)$ for $n=0,1,2,3,\ldots$;(c) also find the n^{th} partial sum of the series in part (a).

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1) a) $\lim_{n\to\infty}\left(\frac{3n-2}{3n+2}\right)^n = \lim_{n\to\infty}\frac{(3n(1-2/3n))^n}{(3n(1+\frac{2}{3n}))^n}$

$= \lim_{n\to\infty}\frac{\left(1-\frac{2}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e^{-2/3}}{e^{2/3}} = e^{-4/3}$ (using $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

b)

$-1 \leq \cos(\ln n) \leq 1$

$-1 \leq (-1)^n\cos(\ln n) \leq 1$

$-1 \leq (-1)^n\cos(\ln n) \leq 1$

as $n\to\infty$ 0 $\quad -\frac{1}{\sqrt{n}} \leq (-1)^n\frac{\cos(\ln n)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ $\quad$ 0 as $n\to\infty$

$\therefore$ by sandwich theorem $\lim_{n\to\infty}\frac{(-1)^n\cos(\ln n)}{\sqrt{n}} = 0$

c) $\lim_{n\to\infty}\frac{1^{-\frac14}+2^{-\frac14}+3^{-\frac14}+\dots+n^{-\frac14}}{n^{3/4}} = \frac{S_n}{n^{3/4}}$

Using Integral Inequality:

$\int_1^{n+1} x^{-\frac14}\,dx \leq S_n < 1+\int_1^n x^{-\frac14}\,dx$

$\left.\frac{4}{3}x^{\frac34}\right|_1^{n+1} < S_n < 1+\left(\left.\frac{4}{3}x^{\frac34}\right|_1^n\right)$

$\frac{4}{3}(n+1)^{\frac34}-\frac43 < S_n < 1+\frac43 n^{3/4}-1$

$\frac{\frac43\left((n+1)^{\frac34}-1\right)}{n^{3/4}} < \frac{S_n}{n^{3/4}} < \frac{\frac43 n^{3/4}}{n^{3/4}}$

$\frac43\left(\frac{(n+1)^{3/4}-1}{n^{3/4}}\right) < \frac{S_n}{n^{3/4}} < \frac43$

The lower bound tense to $\frac{4}{3}$ when $n \to \infty$

since $\lim_{n\to\infty} \frac{(n+1)^{\frac{3}{4}}-1}{n^{3/4}} = 1$.

$\therefore \lim_{n\to\infty} \frac{S_n}{n^{\frac{3}{4}}} = \frac{4}{3}$ by Sandwich-theorem

2) a) $\sum_{n=1}^{\infty} \frac{2+\sin n}{n\sqrt{n}}$

$\sin n \leq 1$

$2+\sin n \leq 3$

$\frac{2+\sin n}{n\sqrt{n}} \leq \frac{3}{n\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{2+\sin n}{n\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} \to$ convergent p-series with $p = 1.5 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{2+\sin n}{n\sqrt{n}}$ is a convergent series by D.C.T

b) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^{\frac{2}{3}}$

Using taylor series,

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \frac{1}{7!\,n^7} + \cdots$

So, $\frac{1}{n} - \sin\frac{1}{n} = \frac{1}{6n^3} - \frac{1}{120n^5} + \frac{1}{7!\,n^7} - \cdots$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{6n^3} - \frac{1}{120n^5} + \frac{1}{7!\,n^7} - \cdots\right)^{2/3}}{\left(\frac{1}{n^3}\right)^{2/3}} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}}\left(\frac{1}{6} - \cancel{\frac{1}{120n^2}}^{0} + \cancel{\frac{1}{7!\,n^4}}^{0} - \cdots\right)^{2/3}}{\cancel{\frac{1}{n^2}}}$$

$= \frac{1}{6}$

$\therefore$ by L.C.T the 2 series behave alike

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$,

then it converges, and so does $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^{2/3}$

c) $\sum_{n=1}^{\infty} \left(e^{\frac{\ln n}{n\sqrt{n}}} - 1\right)$

Same as before, $e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$

Note that we should set $x = \frac{\ln n}{n\sqrt{n}}$ to get our series, but to simplify things we're gonna keep it in x until we simplify the limit.

as $n \to \infty$

$x \to 0$

$$\lim_{x\to 0} \frac{e^{x}-1}{x} = \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{6} + \cdots}{x}$$

$$= \lim_{x\to 0} \frac{x\left(1 + \frac{x}{2} + \frac{x^2}{6} \cdots\right)}{x} = \lim_{x\to 0}\left(1 + \frac{x}{2} + \frac{x^2}{6} + \cdots\right)$$

$$= \lim_{n\to\infty}\left(1 + \cancel{\frac{\ln n}{2n\sqrt{n}}}^{\,0} + \cancel{\frac{\ln^2 n}{6(n\sqrt{n})^2}}^{\,0} + \cdots\right)$$

$$= 1$$

the 2 series, $\sum e^{\frac{\ln n}{n\sqrt{n}}} - 1$ and $\sum \frac{\ln n}{n\sqrt{n}}$

behave alike

$\sum \frac{\ln n}{n\sqrt{n}}$ converges by D.C.T

$\frac{\ln n}{n\sqrt{n}}$ is $\frac{n^{0.1}}{n^{1.5}} = \frac{1}{n^{1.4}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.4}}$ is a convergent p-series $(1.4 > 1)$

$\therefore \sum_{n=1}^{\infty} (e^{\frac{\ln n}{n\sqrt{n}}} - 1)$ converges.

3) $\lim_{n\to\infty} a_n e^{s_n} = 1$

In this problem we'll use this rule

if $a_n \longrightarrow L$

then $f(a_n) \to f(L)$ for any f that's continuous.

Take ln of both sides,

$\ln(\lim_{n\to\infty} a_n e^{s_n}) = \ln 1$

$\lim_{n\to\infty} [\ln(a_n e^{s_n})] = 0$

$\lim_{n\to\infty} (\ln a_n + \ln e^{s_n}) = 0$

$\lim_{n\to\infty} \ln a_n + \lim_{n\to\infty} s_n = 0$

$\lim_{n\to\infty} \ln a_n = -\lim_{n\to\infty} S_n$

take e of both sides

$e^{\lim_{n\to\infty} \ln a_n} = e^{-\lim_{n\to\infty} S_n}$

$\lim_{n\to\infty} e^{\ln a_n} = -\lim_{n\to\infty} e^{-s_n} \Rightarrow \boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{-s_n}}$

Now, if $S_n = \sum a_n$ converges to S

then $\lim_{n\to\infty} a_n = e^{-S}$

Otherwise $S_n \to \infty$

then $\lim_{n\to\infty} a_n = 1$

4) As always, apply ratio test

$$\rho = \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)\ln(n+1)} \quad \frac{|x+3|^{n+1}}{2^{n+1}}}{\frac{1}{n\ln n} \quad \frac{|x+3|^n}{2^n}}$$

$$= \frac{n\ln n \times \cancel{2^n}\, \cancel{|x+3|^n} . |x+3|}{(n+1)\ln(n+1) \times 2 . \cancel{2^n} . \cancel{|x+3|^n}}$$

$$\lim_{n\to\infty} \rho = \lim_{n\to\infty} \cancel{\frac{n}{n+1}}^{1} \times \cancel{\frac{\ln n}{\ln(n+1)}}^{1} . \frac{|x+3|}{2}$$

$$= \frac{|x+3|}{2}$$

for $\rho < 1$

$\frac{|x+3|}{2} < 1$

$|x+3| < 2$

$-2 < x+3 < 2$

$-5 < x < -1$ the series converges

for $x=-1$

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n}\left(\frac{2}{2}\right)^n = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$$

which diverges by integral test,

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln\infty} \frac{du}{u}$$

let $u=\ln x$
$du = \frac{dx}{x}$

$$= \lim_{A\to\infty} \int_{\ln 2}^{A} \frac{du}{u} = \lim_{A\to\infty} \ln u \Big|_{\ln 2}^{\infty}$$

$$= \ln\infty - \ln 2 = \infty$$

for $x=-3$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n} \qquad a_n = \frac{1}{n\ln n}$$

which tense to 0 when $n\to\infty$

a_n is $\searrow$, and $a_n \to 0$

$\therefore$ the series converges by Leibniz theorem.

a_n is $\searrow$ since $f(x) = \frac{1}{x\ln x}$, $f'(x) = \frac{-(\ln x+1)}{(x\ln x)^2} < 0$

but the series diverges absolutely

$\therefore$ at $x=-3$ the series converges conditionally

for $-3<x<-1$ " " " absolutely

a) $f(x) = \frac{3}{4+3x} = \frac{3}{4+3(x-1+1)} = \frac{3}{4+3(x-1)+3}$

$= \frac{3}{7+3(x-1)} = \frac{3}{7(1+\frac{3}{7}(x-1))}$

$= \frac{3}{7} \cdot \frac{1}{1-(-\frac{3}{7}(x-1))} = \frac{3}{7} \sum_{n=0}^{\infty} \left[-\frac{3}{7}(x-1)\right]^n$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{7}\right)^{n+1} (x-1)^n$

b) $f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$

$c_n = \frac{f^n(1)}{n!}$

$\therefore \frac{f^n(1)}{n!} = (-1)^n \left(\frac{3}{7}\right)^{n+1}$

$f^n(1) = n!\,(-1)^n \left(\frac{3}{7}\right)^{n+1}$

c) $S_n = \frac{3}{7} \sum_{n=0}^{\infty} \left(\frac{-3}{7}\right)^n (x-1)^n$

geometric series with $r = \frac{-3}{7}(x-1)$ and $a = \frac{3}{7}$

$= \frac{3}{7}\left(\frac{1-(-\frac{3}{7}(x-1))^n}{1+\frac{3}{7}(x-1)}\right)$

$= \frac{3 - 3\left(\frac{3}{7}(1-x)\right)^n}{7 + 3(x-1)}$