Math 201–Exam 1 (Fall 15)

Prof. B. Shayya and Dr. H. Yamani

- Please write your **section number** on your booklet.
- Please place your student **ID card** on the desk in front of you.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- **Unjustified** answers will receive little or no credit.

Problem 1 (answer on page **1** of the booklet.)
(9 pts each) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(i) $a_n = n(e^{1/n} - 1)$ (ii) $b_n = \sqrt[n]{n}\left(1+\frac{5}{n}\right)^{1/n}\lfloor 3 - 2^{1/n}\rfloor$ (iii) $c_n = \left(2(3^n) + (-1)^n\right)^{1/n}$

Problem 2 (answer on pages **2** and **3** of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3^{n+1}}{5^n} + \frac{(-2)^{n-1}}{3^{n+2}}\right)$ (ii) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^{1.2}}$ (iii) $\displaystyle\sum_{n=2}^{\infty}\frac{3^n}{5^n + n + (-1)^n}$

insight club

Problem 3 (answer on pages **4** and **5** of the booklet.)
Consider the power series

$$F(x) = \sum_{n=0}^{\infty}\left(\sin\frac{1}{n+1}\right)(x-3)^n.$$

(i) (15 pts) Find the radius and interval of convergence of the power series.
(ii) (6 pts) Use ASET to estimate $F(2.9)$ with an error of magnitude less than 10^{-2}. Does you estimate tend to be an over-estimate or an under-estimate.
(iii) (6 pts) Use the geometric series formula to estimate $F(3.1)$ with an error of magnitude less than 10^{-2}.

Problem 4 (answer on page **6** of the booklet.)
(i) (8 pts) State and prove the absolute convergence test (ACT).
(ii) (3 pts) Give an example that shows that the converse of ACT is not true.
(iii) (8 pts) Prove that

$$\left.\begin{array}{l} \sum_{n=1}^{\infty} a_n \quad \text{converges} \\[2ex] \{\sqrt[n]{|a_n|}\} \quad \text{converges} \\[2ex] a_n \neq 0 \quad \text{for all } n \\[2ex] \frac{\ln|a_n|}{n} \not\to 0 \end{array}\right\} \Longrightarrow \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely.}$$

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Problem 1:

i) $a_n = n(e^{1/n}-1)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}-1}{1/n}$$

$$= \lim_{n\to\infty} \frac{-\frac{1}{n^2}e^{1/n}}{-\frac{1}{n^2}} \quad \text{(By L'Hopital Rule)}$$

$$= \lim_{n\to\infty} e^{1/n}$$

$$= e^0$$

$$= 1$$

So, $a_n \to 1$

ii) $b_n = \sqrt[n]{n}\left(1+\frac{5}{n}\right)^{1/n}\left(3-2^{1/n}\right)$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\sqrt[n]{n} \cdot \lim_{n\to\infty}\Big(1+\underbrace{\frac{5}{n}}_{0}\Big)^{\overbrace{1/n}^{0}} \cdot \lim_{n\to\infty}\Big(3-2^{\overbrace{1/n}^{0}}\Big)$$

$$= 1\cdot(1+0)^0\cdot(3-2^0)$$

$$= 2$$

So, $b_n \to 2$

iii) $c_n = \left(2(3^n)+(-1)^n\right)^{1/n}$

$$\left(2(3^n)-1\right)^{1/n} \leq c_n \leq \left(2(3^n)+1\right)^{1/n}$$

$$\lim_{n\to\infty} (2(3^n)-1)^{1/n} = \lim_{n\to\infty} \left(3^n\left(2-\frac{1}{3^n}\right)\right)^{1/n}$$

$$= \lim_{n\to\infty} 3\,\Big(2-\underbrace{\tfrac{1}{3^n}}_{\to 0}\Big)^{\overbrace{1/n}^{\to 0}}$$

$$= 3(2^0)$$

$$= 3$$

similarly, $\lim_{n\to\infty} (2(3^n)+1)^{1/n} = 3$

since $\underbrace{(2(3^n)-1)^{1/n}}_{\downarrow\, 3} \leq c_n \leq \underbrace{(2(3^n)+1)^{1/n}}_{\downarrow\, 3}$

$\Rightarrow c_n \to 3$ by sandwich theorem.

<u>Problem 2:</u>

i) $$\sum_{n=0}^{\infty}\left(\frac{3^{n+1}}{5^n}+\frac{(-2)^{n-1}}{3^{n+2}}\right) = \sum_{n=0}^{\infty}\frac{3^{n+1}}{5^n} + \sum_{n=0}^{\infty}\frac{(-2)^{n-1}}{3^{n+2}}$$

$$= 3\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n + \frac{1}{(-2)(3)^2}\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n$$

$$= 3\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n - \frac{1}{18}\sum_{n=0}^{\infty}\left(\frac{-2}{3}\right)^n$$

$\Rightarrow$ The series converges since it is the sum of two geometric series that have ratios $\left|\frac{3}{5}\right|<1$ & $\left|-\frac{2}{3}\right|<1$.

the sum is $= 3\,\dfrac{1}{1-\frac{3}{5}} - \dfrac{1}{18}\,\dfrac{1}{1-\left(-\frac{2}{3}\right)} = \dfrac{112}{15}$

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}$

$$\frac{\ln n}{n^{1.2}} \Big/ \frac{1}{n^{1.1}} = \frac{\ln n}{n^{0.1}} \longrightarrow 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges since it is p-series with $p=1.1>1$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}$ converges by LCT.

iii) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{5^n+n+(-1)^n}$

$$\frac{3^n}{5^n+n+(-1)^n} < \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$$

$\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{5}\right)^n$ converges since geometric series with $r=\frac{3}{5}<1$

So, $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{5^n+n+(-1)^n}$ converges by DCT.

Problem 3:

i) $F(x) = \sum_{n=0}^{\infty} \left(\sin \frac{1}{n+1}\right)(x-3)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\left(\sin\frac{1}{n+2}\right)(x-3)^{n+1}\right|}{\left|\left(\sin\frac{1}{n+1}\right)(x-3)^n\right|}$$

$$= |x-3| \; \frac{\sin\frac{1}{n+2}}{\sin\frac{1}{n+1}}$$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n+2}}{\sin\frac{1}{n+1}} = \lim_{n\to\infty} \frac{-\frac{1}{(n+2)^2}\cos\frac{1}{n+2}}{-\frac{1}{(n+1)^2}\cos\frac{1}{n+1}}$$ (By L'Hopital Rule)

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2}$$

$$= 1$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \to |x-3|$$

for the series to be convergent, $|x-3| < 1$

$-1 < x-3 < 1$

$2 < x < 4$

for $x=4$: $\sum_{n=0}^{\infty} \sin\frac{1}{n+1}$

$$\lim_{n\to\infty} \sin\frac{1}{n+1} \Big/ \frac{1}{n} = 1$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$) $\Rightarrow \sum_{n=0}^{\infty} \sin\frac{1}{n+1}$ diverges by DCT.

for $x=2$: $\sum_{n=0}^{\infty} (-1)^n \sin\frac{1}{n+1}$

- $\lim_{n\to\infty} \sin\frac{1}{n+1} = 0$
- $\sin\frac{1}{n+1}$ is decreasing.
- $\sin\frac{1}{n+1} > 0$ for all n

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \sin\frac{1}{n+1}$ converges conditionally by AST.

So, interval of convergence: $2 \leq x < 4$

radius of convergence: $R=1$

ii) $F(2.9) = 0.841 - 0.047 + 3.27\times10^{-3} - \cdots$

$\Rightarrow F(2.9) \approx 0.794 + \text{error}$

error $\leq 3.27\times10^{-3} \leq 10^{-2}$ (first unused term)

since error $>0 \Rightarrow$ It is an under-estimate.

iii) $F(3.1) = \sum_{n=0}^{\infty} \sin\left(\frac{1}{n+1}\right)(0.1)^n$

$\left|\sin\left(\frac{1}{n+1}\right)(0.1)^n\right| \leq (0.1)^n$ (Sandwich theorem)

$\downarrow$ 0 $\quad\downarrow$ 0

So, estimate $F(3.1)$ by $\sum_{n=0}^{\infty}(0.1)^n$ which sums to $\frac{1}{1-0.1} = 1.\dot{1}$ $(1.111\ldots)$

for error $< 10^{-2} \Rightarrow$ estimate $F(3.1)$ by 1.11

Problem 4:

i) Absolute Convergence Test (ACT): if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Proof: $-|a_n| \leq a_n \leq |a_n|$

$\Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$

if $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} 2|a_n|$ converges

$\Rightarrow \sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by DCT.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \underbrace{\sum_{n=1}^{\infty} (a_n + |a_n|)}_{\text{converges}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges}}$$

So, $\sum_{n=1}^{\infty} a_n$ converges (sum of 2 convergent series)

ii) converse of ACT is false.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

BUT $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST.

iii) To prove $\sum_{n=1}^{\infty} a_n$ conv absolutely, $\sum_{n=1}^{\infty} |a_n|$ should conv.

- We'll use Root test to prove that:

$$\left\{\sqrt[n]{|a_n|}\right\} \to \rho \quad \text{where } \rho < 1$$

- Given that $\frac{\ln|a_n|}{n} \not\to 0$

$$\Rightarrow \left(e^{\ln|a_n|}\right)^{\frac{1}{n}} \not\to e^{0}$$

$$\sqrt[n]{|a_n|} \not\to 1$$

So, $\rho \neq 1 \Rightarrow$ Root test is NOT inconclusive

- since $\sum_{n=1}^{\infty} a_n$ conv, $a_n \to 0$, so $|a_n| \to 0$

So, ρ is NOT a large number.

$\Rightarrow \rho < 1$.

So, $\left\{\sqrt[n]{|a_n|}\right\} \to \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n|$ conv.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely.