American University of Beirut

الجامعة الأميركية في بيروت

Quiz 1: MATH 201

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Duration: 60 minutes

Name (Last, First):

Student number:

[10 points = 5+5] Problem 1. Determine whether the following sequences are convergent or divergent

(a) $a_n = \dfrac{(\ln n)^{100}}{n^{1.2}}$

as $n \to \infty$, growth of $(\ln n)^{100}$ >>> growth of $n^{1.2}$

So, $a_n = \dfrac{(\ln n)^{100}}{n^{1.2}} \to \infty$

So, a_n diverges

(b) $a_n = (n+4)^{1/(n+4)} \sqrt[n]{4^n n}$

$$\lim_{n\to\infty} a_n = \underbrace{\lim_{n\to\infty} (n+4)^{1/n+4}}_{\downarrow} \cdot \sqrt[n]{4^n} \cdot \sqrt[n]{n}$$

$$\lim_{n\to\infty} (n+4)^{1/n+4} = \lim_{n\to\infty} e^{\frac{1}{n+4}\ln(n+4)}$$

$$= \lim_{n\to\infty} e^{\frac{\frac{1}{n+4}}{1}} \quad \text{... By L'Hopital's Rule}$$

$$= e^0$$

$$= 1$$

So, $\lim_{n\to\infty} a_n = 1 \cdot 4 \cdot 1 = 4$

So, a_n converges to 4.

[10 points] Problem 2. Let $\{a_n\}_n$ be a sequence of real numbers such that $a_n \neq 0$ for all $n \geq 1$. Show that **if** $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n a_n$ is absolutely convergent.

Given that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

$\Rightarrow$ By Root test $\sqrt[n]{|a_n|} \rightarrow \rho < 1$

Applying the same test for $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n a_n$:

$$\sqrt[n]{(1+\frac{1}{n})^n |a_n|} = (1+\frac{1}{n})\sqrt[n]{|a_n|} \rightarrow (1+0)\rho = \rho < 1$$ (Note that $(1+\frac{1}{n}) >> 0$)

So, also $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n a_n$ converges absolutely.

[20 points=10+10] **Problem 3.**

(a) Show that the series $\sum_{n=1}^{\infty}\left[\left(\frac{-2}{3}\right)^n + \frac{2}{(n+2)(n+3)}\right]$ is convergent.

$$\sum_{n=1}^{\infty}\left[\left(-\frac{2}{3}\right)^n + \frac{2}{(n+2)(n+3)}\right] = \sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n + \sum_{n=1}^{\infty}\frac{2}{(n+2)(n+3)}$$

- $\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$ is convergent since it is a geometric series of common ratio $-\frac{2}{3}$ where $\left|-\frac{2}{3}\right| = \frac{2}{3} < 1$

- $\sum_{n=1}^{\infty}\frac{2}{(n+2)(n+3)}$

$$= \sum_{n=1}^{\infty}\frac{\alpha}{n+2} + \frac{\beta}{n+3} \quad (\alpha \,\&\, \beta \text{ are 2 constants})$$

$$\frac{2}{\cancel{(n+2)(n+3)}} = \frac{\alpha(n+3)+\beta(n+2)}{\cancel{(n+2)(n+3)}}$$

$$2 = (\alpha+\beta)n + 3\alpha + 2\beta \Rightarrow \begin{cases}\alpha+\beta=0\\ 3\alpha+2\beta=2\end{cases}$$

$$\Rightarrow \alpha = 2 \;\&\; \beta = -2$$

So, $\sum_{n=1}^{\infty}\frac{2}{(n+2)(n+3)} = \sum_{n=1}^{\infty}\frac{2}{n+2} - \frac{2}{n+3}$

$$= \left(\frac{2}{1+2} - \frac{2}{1+3}\right) + \left(\frac{2}{2+2} - \frac{2}{2+3}\right) + \left(\frac{2}{3+2} - \frac{2}{3+3}\right) + \dots + \left(\frac{2}{n+2} - \frac{2}{n+3}\right)$$

$$= \frac{2}{3} - \frac{2}{n+3} \longrightarrow \frac{2}{3}$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{2}{(n+2)(n+3)}$ is a convergent telescoping series.

So, $\sum_{n=1}^{\infty}\left[\left(\frac{-2}{3}\right)^n + \frac{2}{(n+2)(n+3)}\right]$ converges since it is a sum of two convergent series.

(b) Compute the sum of the series given in part (a) above.

- $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \frac{\left(-\frac{2}{3}\right)}{1-\left(-\frac{2}{3}\right)} = \frac{-2}{5}$

- $\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+3)} = \frac{2}{3} - \frac{2}{n+3} \longrightarrow \frac{2}{3}$

So, $\sum_{n=1}^{\infty} \left[\left(-\frac{2}{3}\right)^n + \frac{2}{(n+2)(n+3)}\right] = \frac{-2}{5} + \frac{2}{3} = \frac{4}{15}$

[20 points: 5 points each] **Problem 4.** Determine whether each of the following series converges or diverges. Give the name of the test you've used and justify your work.

(a) $\sum_{n\geq 1}\left[n\left(\sin\frac{1}{\sqrt{n}}\right)\left(\tan\frac{1}{\sqrt{n}}\right)\right]$

$$n\sin\left(\frac{1}{\sqrt{n}}\right)\tan\left(\frac{1}{\sqrt{n}}\right) = n\,\frac{\sin^2\left(\frac{1}{\sqrt{n}}\right)}{\cos\left(\frac{1}{\sqrt{n}}\right)}$$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \sin\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{n^{1/2}} - \frac{1}{3!\,n^{3/2}} + \frac{1}{5!\,n^{5/2}} - \cdots$
- $\sin^2\left(\frac{1}{\sqrt{n}}\right) = \left(\frac{1}{n^{1/2}} - \frac{1}{3!\,n^{3/2}} + \frac{1}{5!\,n^{5/2}} - \cdots\right)\cdot\left(\frac{1}{n^{1/2}} - \frac{1}{3!\,n^{3/2}} + \frac{1}{5!\,n^{5/2}} - \cdots\right)$

$$= \frac{1}{n} + \left(-\frac{1}{3!} - \frac{1}{3!}\right)\frac{1}{n^2} + \left(\frac{1}{5!} - \frac{1}{3!\,3!} + \frac{1}{5!}\right)\frac{1}{n^3} + \cdots$$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \Rightarrow \cos\left(\frac{1}{\sqrt{n}}\right) = 1 - \frac{1}{2!\,n} + \frac{1}{4!\,n^2} - \cdots$

So, $$n\,\frac{\sin^2\left(\frac{1}{\sqrt{n}}\right)}{\cos\left(\frac{1}{\sqrt{n}}\right)} = \frac{n\left(\frac{1}{n} + \left(-\frac{1}{3!} - \frac{1}{3!}\right)\frac{1}{n^2} + \left(\frac{1}{5!} - \frac{1}{3!\,3!} + \frac{1}{5!}\right)\frac{1}{n^3} + \cdots\right)}{1 - \frac{1}{2!\,n} + \frac{1}{4!\,n^2} - \cdots}$$

$$= \frac{1 + \left(-\frac{1}{3!} - \frac{1}{3!}\right)\frac{1}{n} + \left(\frac{1}{5!} - \frac{1}{3!\,3!} + \frac{1}{5!}\right)\frac{1}{n^2} + \cdots}{1 - \frac{1}{2!\,n} + \frac{1}{4!\,n^2}} \to 1 (\neq 0)$$

as $n \to \infty$

So, $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{\sqrt{n}}\right)\tan\left(\frac{1}{\sqrt{n}}\right)$ diverges by n^{th} term test.

(b) $\sum_{n\geq 1} \frac{n+\ln n}{n^{\frac{5}{2}}+n^{\frac{1}{2}}+100}$

$$\frac{n+\ln n}{n^{5/2}+n^{1/2}+100} < \frac{n+\ln n}{n^{5/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p=\frac{3}{2}>1$)

and $\lim_{n\to\infty} \frac{n+\ln n}{n^{5/2}} \Big/ \frac{1}{n^{3/2}} = \lim_{n\to\infty} \frac{n+\ln n}{n} = \lim_{n\to\infty} 1+\frac{\ln n}{n} = 1+0=1$

(which is a +ve constant)

So, Both series behave similarly $\Rightarrow \sum_{n=1}^{\infty} \frac{n+\ln n}{n^{5/2}}$ converges by LCT

So, $\sum_{n=1}^{\infty} \frac{n+\ln n}{n^{5/2}+n^{1/2}+100}$ converges by DCT

(c) $\sum_{n\geq 1} \frac{1}{\sqrt{n}(1+\sqrt{n})}$

Take $f(x) = \frac{1}{\sqrt{x}(1+\sqrt{x})}$

- $\sqrt{x}(1+\sqrt{x})$ is increasing $\Rightarrow$ $f(x) = \frac{1}{\sqrt{x}(1+\sqrt{x})}$ is decreasing
- $f(x)$ is positive and continuous for $x \geq 1$.

$\Rightarrow \int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{\sqrt{x}(1+\sqrt{x})}$ (let $u = 1+\sqrt{x} \Rightarrow 2du = \frac{dx}{\sqrt{x}}$)

$= \int^\infty \frac{2du}{u}$

$= [2\ln|u|]_1^\infty$

$= [2\ln(1+\sqrt{x})]_1^\infty$

$= \infty$ $\Rightarrow$ diverges by integral test.

(d) $\sum_{n\geq 1} \left[\frac{1}{6-\left(1+\frac{1}{n}\right)^n}\right]^n$

$\lim \sqrt[n]{\left[\frac{1}{6-(1+\frac{1}{n})^n}\right]^n} = \lim \frac{1}{6-(1+\frac{1}{n})^n} = \frac{1}{6-e} < 1$

$\Rightarrow$ converges by Root Test.

[10 points] **Problem 5.** For what values of x does the power series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2^n \ln n} x^n$ converge?

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{2^n \ln n} x^n \right| = \sum_{n=2}^{\infty} \frac{|x|^n}{2^n \ln n}$$

By Ratio test: $\lim \frac{|x|^{n+1}}{2^{n+1} \ln(n+1)} \cdot \frac{2^n \ln n}{|x|^n}$

$$= \lim |x| \cdot \frac{1}{2} \cdot \frac{\ln n}{\ln(n+1)}$$

$$= \frac{1}{2}|x|$$

for the series to be convergent: $\frac{1}{2}|x| < 1$

$$-1 < \frac{1}{2}x < 1$$

$$-2 < x < 2$$

for $x=-2$: $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(-1)^n \cancel{(+2)^n}}{\cancel{2^n} \ln n} = \sum_{n=2}^{\infty} -\frac{1}{\ln n}$

$$\ln n < n$$

$$\frac{1}{\ln n} > \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since p-series and $p=1$.

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by DCT

So, $\sum_{n=2}^{\infty} -\frac{1}{\ln n}$ diverges for $x=2$: $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$

- $\ln n$ is increasing $\Rightarrow \frac{1}{\ln n}$ is decreasing
- $\frac{1}{\ln n} > 0$ for $n \geq 2$
- $\frac{1}{\ln n} \to 0$

$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$ converges conditionally by AST.

So, $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n}{2^n \ln n}$ converges for $-2 < x \leq 2$.

[20 points] Problem 6. Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n(n!)}$. For each positive integer n, let S_n denote the sum of the first n terms of the given series.

(a) (**10 points**) Explain why the given series is convergent.

Take the absolute value of the series: $\sum_{n=1}^{\infty} \frac{1}{2^n(n!)}$

$$\lim \frac{1}{2^{n+1}(n!)(n+1)} \cdot 2^n(n!) = \lim \frac{1}{2(n+1)} = 0 < 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n(n!)}$ converges by Ratio test.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n(n!)}$ converges by ACT.

(b) (**5 points**) Let S be the sum of the given series. Find the least value of n so that S_n approximates S with an absolute error less than or equal to 10^{-4}.

$|\text{error}| = |S - S_n| \leq 10^{-4}$

$|\text{the fourth term}| = |-2.6 \times 10^{-3}| = 2.6 \times 10^{-3} > 10^{-4}$

$|\text{the fifth term}| = |2.6 \times 10^{-4}| = 2.6 \times 10^{-4} > 10^{-4}$

$|\text{the sixth term}| = |-2.17 \times 10^{-5}| = 2.17 \times 10^{-5} < 10^{-4}$

$\Rightarrow$ S is approximated by S_5

$\Rightarrow$ the least value of n is 5.

(c) (**5 points**) Is the partial sum obtained in part (b) an under estimation of S? Justify.

the first unused term (sixth term) $= -2.17 \times 10^{-5}$ which is negative

$\Rightarrow$ the partial sum is over estimation of S NOT under estimation.

[10 points] **Problem 7.** Find the Taylor polynomial of order 7, about $x = 0$, for the function

$$f(x) = \sqrt{1+x}.$$

- $f(x) = \sqrt{1+x} = (1+x)^{1/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

$f^{IV}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}} \Rightarrow f^{IV}(0) = \frac{-15}{16}$

$f^{V}(x) = \frac{105}{32}(1+x)^{-\frac{9}{2}} \Rightarrow f^{V}(0) = \frac{105}{32}$

$f^{VI}(x) = -\frac{945}{64}(1+x)^{-\frac{11}{2}} \Rightarrow f^{VI}(0) = \frac{-945}{64}$

$f^{VII}(x) = \frac{10395}{128}(1+x)^{-\frac{13}{2}} \Rightarrow f^{VII}(0) = \frac{10395}{128}$

- Taylor's series of $f(x) = \sqrt{1+x}$ about $x=0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = \frac{1\cdot x^0}{0!} + \frac{\frac{1}{2}\cdot x}{1!} + \frac{\frac{-1}{4}\cdot x^2}{2!} + \frac{\frac{3}{8}\cdot x^3}{3!} + \frac{\frac{-15}{16}\cdot x^4}{4!} + \frac{\frac{105}{32}\cdot x^5}{5!}$$

$$+ \frac{-945\cdot x^6}{64\cdot 6!} + \frac{\frac{10395}{128}\cdot x^7}{7!} + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$$

$$-\frac{21}{1024}x^6 + \frac{33}{2048}x^7 + \cdots$$

- $P_7(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \frac{33}{2048}x^7$

10