Math 201 Exam 1 (Fall 2013) **Dr. H. Yamani**

- **Please write your section number on your booklet**
- **Please answer each problem on the indicated page(s) of the booklet. Any part of your answer that is not written on the indicated page(s) will not be graded.**
- **Unjustified answers will receive little or no credit.**

Problem 1 (answer on pages 1 and 2 of the booklet) . (7 points each)
Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence

(i) $a_n = \left(n^{1/\ln n} + \dfrac{0.6^n}{n!} + n^{3/n}\right)$ (ii) $c_n = \left(\dfrac{n+\ln 5}{n+\ln 6}\right)^{n+1}$

(iii) $b_n = 2^{\frac{(-1)^n}{n}}$ (iv) $d_n = n\left(7^{1/n} - 1\right)$

Problem 2 (answer on pages 3 and 4 of the booklet). (7 points each)
Which of the following series converge, and which diverge?
Find the sum of the series if possible.

(i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2^n} + \frac{2^n}{7^n}\right)$ (ii) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}} - \sin\left(\frac{1}{\sqrt{n}}\right)\right]$

(iii) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt[n]{n} + 1\right)$ (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{0.1}\left(e^{2n}+1\right)}$

$\sin x \approx x - \frac{x^3}{3}$

$-\sin\left(\frac{1}{\sqrt{n}}\right)$

$\frac{1}{\sqrt{n}}$

$\sin x - x \approx$

$(x - \sin x) \approx$

$\dfrac{\left(\frac{1}{\sqrt{n}}\right)^3}{3!}$

Problem 3 (answer on page 5 of the booklet) (16 points)
Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{\left(\sqrt{n}+1\right)3^n}$$

For what values of x does the series converge absolutely? Conditionally?

Problem 4 (answer on page 6 of the booklet)

Consider the function $f(x) = \dfrac{1}{(1-x)^2}$

$(1-x)^{-2}$

$-2(1-x)^{-3}$

$6(1-x)^{-4}$

$-24(1-x)^{-5}$

(i) (7 pts) Prove that $f(x) = \displaystyle\sum_{n=1}^{\infty} n x^{n-1}$ $\quad -1 < x < 1$

Also find the Taylor polynomials $p_2(x)$ and $p_3(x)$ generated by f at $x = 0$.

(ii) (7 pts) Approximate $f(-0.02)$ by $p_2(-0.02)$ and estimate the resulting error using the alternating series estimation theorem.

(iii) (7 pts) Approximate $f(0.1)$ by $p_2(0.1)$ and estimate the resulting error using Taylor's Theorem

Problem 5 (answer on last page of the booklet)
State and prove the theorem of the Direct Comparison Test for nonnegative series. (7 pts)

Yamani / Fall 2013-2014 / Quiz I (Suggested Solutions)

Problem 1:

i) $a_n = n^{1/\ln n} + \dfrac{0.6^n}{n!} + n^{3/n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[e^{\ln(n^{1/\ln n})} + \frac{0.6^n}{n!} + (n^{1/n})^3\right]$$

$$= \lim_{n\to\infty}\left[e^{\frac{\cancel{\ln n}}{\cancel{\ln n}}} + \frac{0.6^n}{n!} + (n^{1/n})^3\right]$$

since growth of factorial >>> growth of power

$\Rightarrow n! >>> 0.6^n$

$\Rightarrow \dfrac{0.6^n}{n!} \to 0$

$$= e + 0 + 1^3$$
$$= e + 1$$

$$\boxed{\Rightarrow a_n \to e+1}$$

ii) $c_n = \left(\dfrac{n+\ln 5}{n+\ln 6}\right)^{n+1}$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\left(\frac{n+1-1+\ln 5}{n+1-1+\ln 6}\right)^{n+1}$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{n+1}}{\cancel{n+1}}\right)^{n+1}\left(\frac{1+\frac{-1+\ln 5}{n+1}}{1+\frac{-1+\ln 6}{n+1}}\right)^{n+1}$$

$$= \frac{e^{-1+\ln 5}}{e^{-1+\ln 6}}$$

$$= \frac{5}{6} \quad \boxed{\Rightarrow c_n \to \frac{5}{6}}$$

iii) $b_n = 2^{(-1)^n/n}$

$$\lim_{n\to\infty} b_n = 2^{\lim_{n\to\infty}(-1)^n/n} = 2^0 = 1$$

$$\boxed{\Rightarrow b_n \to 1}$$

iv) $d_n = n(7^{1/n} - 1)$

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{7^{1/n}-1}{1/n}$$

$$= \lim_{n\to\infty} \frac{(\ln 7)\cancel{\left(-\frac{1}{n^2}\right)} 7^{1/n}}{\cancel{\left(-\frac{1}{n^2}\right)}} \quad \text{... By LHopital's}$$

$$=(\ln 7)(7^0)$$

$$= \ln 7$$

$$\boxed{\Rightarrow d_n \to \ln 7}$$

Problem 2:

i) $\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^n} + \frac{2^n}{7^n}\right]$

$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$ ... converges since sum of two geometric series.

the sum of the series $= \frac{1}{1-(-\frac{1}{2})} + \frac{1}{1-\frac{2}{7}}$

$= \frac{31}{15}$

ii) $\sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n}} - \sin\left(\frac{1}{\sqrt{n}}\right)\right]$

$$\sin\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} - \frac{\left(\frac{1}{\sqrt{n}}\right)^3}{3!} + \frac{\left(\frac{1}{\sqrt{n}}\right)^5}{5!} - \frac{\left(\frac{1}{\sqrt{n}}\right)^7}{7!} + \cdots$$

(we used the fact that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$)

$$\frac{1}{\sqrt{n}} - \sin\left(\frac{1}{\sqrt{n}}\right) = \frac{\left(\frac{1}{\sqrt{n}}\right)^3}{3!} - \frac{\left(\frac{1}{\sqrt{n}}\right)^5}{5!} + \frac{\left(\frac{1}{\sqrt{n}}\right)^7}{7!} + \cdots$$

$$= \frac{1}{3!\,n^{3/2}} - \frac{1}{5!\,n^{5/2}} + \frac{1}{7!\,n^{7/2}} + \cdots$$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\frac{1}{3!\,n^{3/2}} - \frac{1}{5!\,n^{5/2}} + \frac{1}{7!\,n^{7/2}} - \cdots}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{1}{3!} - \frac{1}{5!\,n} + \frac{1}{7!\,n^2} \cdots$$

$$= \frac{1}{3!}$$

$$= \frac{1}{6}$$

since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p>1$)

So, $\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right]$ converges by LCT

iii) $\sum_{n=1}^{\infty} \sqrt[n]{n}+1$

$\lim_{n\to\infty} \sqrt[n]{n}+1 = 1+1 = 2 \neq 0$

So, $\sum_{n=1}^{\infty} \sqrt[n]{n}+1$ diverges by n^{th} term test

iv) $\sum_{n=1}^{\infty} \frac{1}{n^{0.1}(e^{2n}+1)}$

By Ratio test: $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^{0.1}(e^{2(n+1)}+1)}}{\frac{1}{n^{0.1}(e^{2n}+1)}}$

$$= \lim_{n\to\infty} \frac{n^{0.1}}{(n+1)^{0.1}} \frac{e^{2n}+1}{e^{2(n+1)}+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{2}e^{2n}}{\cancel{2}e^2 e^{2n}} \quad \cdots \text{ By L'Hopital's}$$

$$= \frac{1}{e^2} < 1$$

So, $\sum_{n=1}^{\infty} \frac{1}{n^{.1}(e^{2n}+1)}$ converges by Ratio test.

Problem 3:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{(\sqrt{n+1})3^n} \quad \cdots \quad \text{let } a_n = \frac{(-1)^n (x-1)^n}{(\sqrt{n+1})3^n}$$

By Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{3^{n+1}\sqrt{n+2}} \cdot \frac{\sqrt{n+1}\,3^n}{(x-1)^n(-1)^n} \right|$

$$= \lim_{n\to\infty} \frac{1}{3} \frac{\sqrt{n+1}}{\sqrt{n+2}} |x-1|$$

$$= \frac{1}{3}|x-1|$$

To converge, $\frac{1}{3}|x-1| < 1$

$$|x-1| < 3$$

$$-3 < x-1 < 3$$

$$-2 < x < 4$$

for $x=-2$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{\sqrt{n+1}\;3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

$\lim \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = 1$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p<1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges by LCT for $x=4$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ where $b_n = \frac{1}{\sqrt{n+1}}$, $b_n > 0$

b_n is decreasing

$b_n \to 0$

So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by AST.

So, $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{(\sqrt{n+1})\, 3^n}$ converges absolutely for $-2 < x < 4$

& converges conditionally for $x=4$

Problem 4:

$f(x) = (1-x)^{-2}$

i) $f(x) = (1-x)^{-2}$, $f(0)=1$

$f'(x) = 2(1-x)^{-3}$, $f'(0)=2$

$f''(x) = 6(1-x)^{-4}$, $f''(0)=6$

$f'''(x) = 24(1-x)^{-5}$, $f'''(0)=24$

$\vdots$ $\vdots$

Using the fact that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

According to Taylor's and taking $a=0$:

$$f(x) = 1 + 2x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=1}^{\infty} n x^{n-1} \quad (-1 < x < 1)$$

$P_2(x) = 1 + 2x$

$P_3(x) = 1 + 2x + 3x^2$

ii) $P_2(-0.02) = 1 + 2(-0.02) = 0.96 \Rightarrow f(-0.02) = 0.96$ if estimated by $P_2(x)$

By using ASET, $|error| \leqslant |$first unused term$|$

$|error| \leqslant |3(-0.02)^2|$

$\Rightarrow |error| \leqslant 0.0012$

iii) $f(0.1) = P_2(0.1) + R_2(0.1)$

$P_2(0.1) = 1 + 2(0.1) = 1.2$

error $\simeq R_2(0.1) = 3(0.1)^2 = 0.03$

Problem 5:

DCT: $\sum a_n, \sum b_n, \sum c_n$ are series with nonnegative terms.

for some integer N: $b_n \le a_n \le c_n$ for $n > N$.

So, a) if $\sum c_n$ converges $\Rightarrow$ $\sum a_n$ converges.

b) if $\sum b_n$ diverges $\Rightarrow$ $\sum a_n$ diverges.

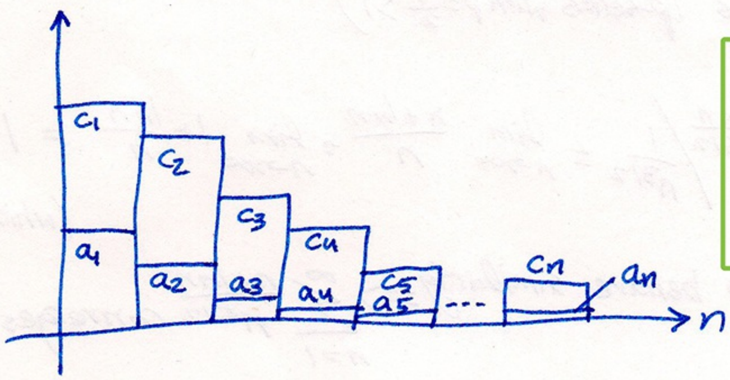
Proof: a) partial sums of $\sum a_n$ are bounded above by:

$$M = a_1 + a_2 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n$$

$\Rightarrow$ they form a nondecreasing sequence of limit $L \le M$

$\Rightarrow$ if $\sum c_n$ converges, $\sum a_n$ converges.

In the following figure, each term is interpreted as area of a rectangle.



b) partial sums of $\sum a_n$ are not bounded from above,

if they were, the partial sums of $\sum b_n$ would be bounded by:

$$M' = b_1 + b_2 + \cdots + b_N + \sum_{n=N+1}^{\infty} a_n$$

$\Rightarrow$ since $\sum a_n$ diverges, $\sum b_n$ diverges.