American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2012

quiz # 1

Exercise 1 *(10 points)* Find the limit of the following **sequences**:

a) $\dfrac{n^2}{2n+1}\sin(3/n)$ b) $\dfrac{n^n+1}{2^n+n!}$ c) $\left(1+\frac{1}{3n}\right)^{2n}$

Exercise 2 *(35 points)* Determine if the following **series** converges or diverges **Justify your answers**

a) $\displaystyle\sum_{n=1}^{+\infty}\frac{10^n}{(\ln n)^n}$

b) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n2^n-1}$

c) $\displaystyle\sum_{n=1}^{+\infty}\frac{2\cos(n!)-1}{n(n+1)}$

d) $\displaystyle\sum_{n=2}^{+\infty}\frac{\ln(1+e^{3n^2})}{n\sqrt{n}}$

e) $\displaystyle\sum_{n=1}^{+\infty}(e^{2/n}-1)$

Exercise 3 *(20 points)* a) Find the interval of convergence of the power series

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n4^n}(3x-1)^{2n}$$

(do not forget to check at the end points)

b) For what value(s) of x the series converges absolutely ? conditionally ?

Exercise 4 *(15 points)* Let $f(x)=\dfrac{x-1}{3+2x}$. Find the Taylor series of f about $x=1$, then find $f^{(101)}(1)$

Exercise 5 *(10 points)* Find the following limit: $\displaystyle\lim_{x\to 0}\frac{\cos(\sqrt{x})-1+\frac{x}{2}}{3x^2}$

Exercise 6 *(10 points)* By using the Maclaurin series of $\ln(1+x)$, give an estimate of $\ln(1.1)$ with an error of magnitude less than 10^{-3}

quiz1 solution

1) a) $\frac{n^2}{2n+1}\sin\left(\frac{3}{n}\right) = \frac{n}{2n+1}\times\frac{\sin\left(\frac{3}{n}\right)}{\frac{1}{n}} \xrightarrow[\infty]{} \frac{1}{2}\times 3 = \frac{3}{2}$

b) $\frac{n^n+1}{2^n+n!} = \frac{n^n}{n!}\left(\frac{1+\frac{1}{n^n}}{\frac{2^n}{n!}+1}\right) \xrightarrow[\infty]{} +\infty$

c) $\left(1+\frac{1}{3n}\right)^{2n} = \left(\left(1+\frac{1/3}{n}\right)^n\right)^2 \xrightarrow[\infty]{} \left(e^{1/3}\right)^2 = e^{2/3}$

2) a) $\sqrt[n]{\frac{10^n}{\ln^n n}} = \frac{10}{\ln n} \xrightarrow[\infty]{} 0 \Rightarrow \sum\left(\frac{10}{\ln n}\right)^n$ cv. by root test

b) $\sum_{n=1}^{+\infty}\frac{1}{n2^n-1}$ cv. by LCT with $\sum_{n=1}^{+\infty}\frac{1}{2^n}$

c) $\frac{3+\cos(n!)}{n(n+1)} \leqslant \frac{4}{n^2}$

but $\sum\frac{4}{n^2}$ cv. (p-series with $p=2$)

$\Rightarrow \sum\frac{3+\cos(n!)}{n(n+1)}$ cv. by DCT

d) $\frac{\ln(1+e^{3n^2})}{n\sqrt{n}} \xrightarrow[n\to+\infty]{} +\infty$ (why !!)

then the series $\sum\frac{\ln(1+e^{3n^2})}{n\sqrt{n}}$ div. by nth term test

e) $e^{2/n}-1 = \left(1+\frac{2}{n}+\frac{2}{n^2}+\dots\right)-1 = \frac{2}{n}+\frac{2}{n^2}+o\left(\frac{1}{n^2}\right)$

$\Rightarrow \frac{e^{2/n}-1}{\frac{1}{n}} = 2+\frac{2}{n}+o\left(\frac{1}{n}\right) \xrightarrow[\infty]{} 2$

Thus $\sum(e^{2/n}-1)$ and $\sum\frac{1}{n}$ have the same nature (by LCT), thus diverges

3) $\sqrt[n]{\left|\frac{(-1)^{n-1}}{n4^n}(3x-1)^{2n}\right|} = \frac{1}{\sqrt[n]{n}\times 4}\times(3x-1)^2 \longrightarrow \frac{(3x-1)^2}{4}$

if $\frac{(3x-1)^2}{4}<1 \Rightarrow (3x-1)^2<4$

$\Rightarrow -2<3x-1<2$

$\Rightarrow \boxed{-1/3<x<1}$

the series cv. abs.

at $x=-1/3$

$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n4^n}\times 4^n = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}$

cv. by AST

at $x=1$

$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}$ cv. by AST

$\boxed{IC = -\frac{1}{3}\leqslant x\leqslant 1}$

abs. cv.: $-1/3<x<1$

cond. cv.: $\{-1/3, 1\}$

4) Let $u = x-1$

$f = \frac{u}{3+2(u+1)} = \frac{u}{5}\times\frac{1}{1+\frac{2}{5}u}$

$= \frac{u}{5}\times\sum_{n=0}^{+\infty}(-1)^n\left(\frac{2}{5}\right)^n u^n$

$\Rightarrow f(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{2^n}{5^{n+1}}(x-1)^{n+1}$

b) $f(x) = \sum_{n=0}^{+\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$

by identification: $\frac{f^{(101)}(1)}{(101)!} = (-1)^{100}\frac{2^{100}}{5^{101}}$

$\rightarrow f^{(101)}(1) = (101)!\times\frac{2^{100}}{5^{101}}$

5) $\lim_{x\to 0}\frac{\cos(\sqrt{x})-1+x/2}{3x^2}$

$= \lim_{x\to 0}\frac{\cancel{1}-\cancel{x/2}+\frac{x^2}{4!}+o(x^2)-\cancel{1}+\cancel{x/2}}{3x^2}$

$= \lim_{x\to 0}\frac{1}{72}+o(x^2) = \frac{1}{72}$

6) $\ln(1+x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{n+1}$

$-1<x\leqslant 1$

as the series cv. at $x=0.1$

$\Rightarrow \ln(1.1) = \sum_{n=0}^{+\infty}(-1)^n\frac{(0.1)^{n+1}}{n+1}$

$= 0.1-\frac{(0.1)^2}{2}+\underbrace{\frac{(0.1)^3}{3}}_{<10^{-3}}+\dots$

$\Rightarrow$ by ASET

$\ln(1.1)\simeq 0.1-\frac{(0.1)^2}{2}$