CMPS 251
Practice Problems-Solution

Problem 1

First using Lagrange polynomial

$$l_0(t) = \frac{(t-1)(t-4)}{4}$$

$$l_1(t) = -\frac{t(t-4)}{3}$$

$$l_3(t) = \frac{t(t-1)}{12}$$

Hence $p_2(t) = l_0(t) + 2l_1(t) + 2l_2(t)$

Expanding then simplifying, we get $p_2(t) = -\frac{1}{4}t^2 + \frac{5}{4}t + 1$

Now using monomials, we form the Vandermonde matrix and solve the system Ac=y

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Solving the system, we get $c_0 = 1, c_1 = \frac{5}{4}, c_2 = -\frac{1}{4}$

Problem 2

a) $f(x) = \sqrt{x}, x_i = 0,1,4$

$$f[x_0] = 0, f[x_1] = 1, f[x_2] = 2;$$
$$f[x_0, x_1] = \frac{1}{1} = 1, f[x_1, x_2] = \frac{2-1}{4-1} = \frac{1}{3}$$
$$f[x_0, x_1, x_2] = (f[x_1, x_2] - \frac{f[x_0, x_1])}{x_2 - x_0} = \frac{\frac{1}{3} - 1}{4} = -\frac{1}{6}$$

b) $f(x) = \ln x, x_i = 1, \frac{3}{2}, 2.$

We get the following values

	$f[\cdot]$	$f[,]$	$f[,,]$
x_0	0	0.8109	-0.2356
x_1	0.4055	0.5754	
x_2	0.6931		

c) $f(x) = \sin \pi x, x_i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

	$f[\cdot]$	$f[,]$	$f[,,]$	$f[,,,]$	$f[,,,,]$
x_0	0	2.8284	-3.3137	-1.8301	3.6602
x_1	0.7071	1.1716	-4.6863	1.8301	
x_2	1.0000	-1.1716	-3.3137		
x_3	0.7071	-2.8284			
x_4	0.0000				

d) $f(x) = \log_2 x, x_i = 1,2,4$

	$f[\cdot]$	$f[,]$	$f[,,]$
x_0	0	1	-0.1667
x_1	1	0.5	
x_2	2		

e) $f(x) = \sin \pi x, x_i = -1,0,1$

	$f[\cdot]$	$f[,]$	$f[,,]$
x_0	0	0	0
x_1	0	0	
x_2	0		

Problem 3

We can use Theorem 2 for this problem since we have equal spacing of the points. A quadratic polynomial indicates that n=2.

For $x \in \left[\frac{1}{4}\ \ 1\right], |f'''(x)| = \frac{3}{8}x^{-\frac{5}{2}} < M = \frac{3}{8} \cdot 4^{\frac{5}{2}}$

$$|f(x) - p(x)| < \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{3}{8} \cdot 4^{\frac{5}{2}} \cdot \left(\frac{\left(1 - \frac{1}{4}\right)}{2}\right)^3 = 0.2953$$

For the quartic polynomial, n=4, , $\left|f^{(5)}(x)\right| = \frac{105}{32}x^{-\frac{9}{2}} < M = \frac{105}{32} \cdot 4^{\frac{9}{2}}$

$$|f(x) - p(x)| < \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{105}{32} \cdot 4^{\frac{9}{2}} \cdot \left(\frac{\left(1 - \frac{1}{4}\right)}{4}\right)^5 = 0.0195$$

Problem 4

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) + \frac{h^5}{120}f^{(5)}(x_0) + \frac{h^6}{720}f^{(6)}(x_0) + \cdots$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) - \frac{h^5}{120}f^{(5)}(x_0) + \frac{h^6}{720}f^{(6)}(x_0) + \cdots$$

$$f(x_0+h)+f(x_0-h)=2f(x_0)+h^2f''(x_0)+\frac{h^4}{12}f^{(4)}(x_0)+O(h^6)$$

Thus $f''(x_0)=\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}-\frac{h^2}{12}f^{(4)}(x_0)+O(h^4)$

So $c=-\frac{1}{12}f^{(4)}(x_0)$

Problem 5

We can use Lagrange interpolation to find the polynomial

$$p_2(x)=8\frac{(x-2)(x-4)}{(1-2)(1-4)}+4\frac{(x-1)(x-4)}{(2-1)(2-4)}+2\frac{(x-1)(x-2)}{(4-1)(4-2)}=x^2-7x+14$$

$$f(3)=\frac{8}{3},\qquad p(3)=2$$

The true error is therefore $\left|\frac{8}{3}-2\right|=2/3$

Now using Theorem 1 with $f'''(\xi)<48$ on the interval $[1\quad 4]$

$$|f(3)-p(3)|<\left|\frac{1}{3!}\cdot 48\cdot(3-1)(3-2)(3-4)\right|=16$$

We can see that this error bound is very loose, compared to the true error

Problem 6

Three point backward difference (h=0.2)

$$f''(1.4)=\frac{f(1)-2f(1.2)+f(1.4)}{(0.2)^2}=-0.1225$$

Three point forward difference (h=0.2)

$$f''(1.4)=\frac{f(1.4)-2f(1.6)+f(1.8)}{(0.2)^2}=-0.033$$

Three point central difference (h=0.2)

$$f''(1.4)=\frac{f(1.2)-2f(1.4)+f(1.6)}{(0.2)^2}=-0.0706$$

Three point central difference (h=0.1)

$$f''(1.4)=\frac{f(1.3)-2f(1.4)+f(1.5)}{(0.1)^2}=-0.0698$$

For comparison, the exact value is -0.0695

Problem 7

Let $x_0 = 2, x_1 = 2.5, x_2 = 4$

We are required to find S_{22}

$$S_{00}(x) = f(x_0) = 0.5, \qquad S_{10}(x) = 0.4 \qquad S_{20}(x) = 0.25$$

$$S_{11}(x) = \frac{(x-2)}{0.5}0.4 + \frac{(2.5-x)}{0.5}0.5 = \frac{9}{10} - \frac{x}{5}$$

$$S_{21}(x) = \frac{(x-2.5)}{1.5}0.25 + \frac{(4-x)}{1.5}0.4 = \frac{13}{20} - \frac{x}{10}$$

Finally $S_{22}(x) = \frac{(x-2)}{2}\left(\frac{13}{20} - \frac{x}{10}\right) + \frac{(4-x)}{2}\left(\frac{9}{10} - \frac{x}{5}\right) = \frac{x^2}{20} - \frac{17x}{40} + \frac{23}{20}$

Problem 8

This is an example of inverse interpolation, where the roles of x and y are interchanged. Instead of computing y at a given x, we are finding x that corresponds to a given y (in this case, y = 0). Applying Neville's method, you should get the root x=3.8317

Problem 9

We can use the central difference formula
for h=0.1

$$f''(1.3) = \frac{f(1.2) - 2f(1.3) + f(1.4)}{0.1^2} = 36.641$$

for h=0.01

$$f''(1.3) = \frac{f(1.29) - 2f(1.3) + f(1.31)}{0.01^2} = 36.5$$

The exact value is 36.5935

Problem 10

$$f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(\xi_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h}$$
$$= \frac{e(x_0+h) - e(x_0)}{h} - \frac{h}{2}f''(\xi_0)$$

Since the error is bounded by ϵ

$$\left|\frac{e(x_0+h) - e(x_0)}{h} - \frac{h}{2}f''(\xi_0)\right| \le \frac{2\epsilon}{h} + \frac{hM}{2}$$

where $|f''(\xi)| < M$ in the given interval

Problem 11

$$D(1,1) = \frac{4}{3}D(1,0) - \frac{1}{3}D(0,0)$$

$$D(1,0) = \phi\left(\frac{h}{2}\right) = \phi(0.25)$$
$$D(0,0) = \phi(h) = \phi(0.5)$$

Using a central difference equation and x=0.5

$$D(1,0) = \frac{f(0.75) - f(0.25)}{2 * 0.25} = \frac{0.6363 - 1.1035}{0.5} = -0.9344$$
$$D(0,0) = \frac{f(1) - f(0)}{2 * 0.5} = \frac{0.2 - 1.2}{1} = -1$$

Hence

$$D(1,1) = \frac{4}{3} \cdot -0.9344 - \frac{1}{3} \cdot -1 = -0.9126$$

Problem 12

For this problem we use Lagrange interpolation

First note that we have two distinct values of x and y

$$l_0(x) = 1 - x$$
$$l_1(x) = x$$
$$\overline{l_0}(y) = 1 - y$$
$$\overline{l_1}(y) = y$$

The resulting polynomial is

$$5l_0(x)\overline{l_0}(y) + 4l_0(x)\overline{l_1}(y) + 3l_1(x)\overline{l_0}(y) + 6l_1(x)\overline{l_1}(y)$$
$$= 5(1 - x)(1 - y) + 4(1 - x)y + 3x(1 - y) + 6xy$$
$$= 4xy - 2x - y + 5$$

Problem 13

For this problem, one can either derive the Newton polynomial as was done in class or using the Lagrange polynomial

Using the Lagrange polynomial

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$
$$= \frac{(x - x_1)(x - x_2)}{h_1(h_1 + h_2)} f(x_0) - \frac{(x - x_0)(x - x_2)}{h_1 h_2} f(x_1) + \frac{(x - x_0)(x - x_1)}{(h_1 + h_2)h_2} f(x_2)$$

Now taking the derivative, we have

$$p_2'(x) = \frac{f(x_0)}{h_1(h_1 + h_2)}(2x - x_1 - x_2) - \frac{f(x_1)}{h_1 h_2}(2x - x_0 - x_2) + \frac{f(x_2)}{(h_1 + h_2)h_2}(2x - x_0 - x_1)$$

Now at x_1

$$p_2'(x_1) = \frac{f(x_0)}{h_1(h_1 + h_2)}(x_1 - x_2) - \frac{f(x_1)}{h_1 h_2}(2x_1 - x_0 - x_2) + \frac{f(x_2)}{(h_1 + h_2)h_2}(x_1 - x_0)$$
$$p_2'(x_1) = \frac{f(x_0)}{h_1(h_1 + h_2)}(-h_2) - \frac{f(x_1)}{h_1 h_2}(h_1 - h_2) + \frac{f(x_2)}{(h_1 + h_2)h_2}(h_1)$$

When $h_1 = h_2 = h$, the second term disappears and we get

$$p_2'(x_1) = \frac{f(x_0)}{2h^2}(-h) + \frac{f(x_2)}{2h^2}(h) = \frac{f(x_2) - f(x_0)}{2h}$$

This is the two point central difference formula!