Math 201 Exam 1 (Spring 2012) B. Shayya and H. Yamani

- Please write your name and section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet.

Any part of your answer that is not written on the indicated page(s) will not be graded.

- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1 and 2 of the booklet.)

(8 pts each) Which of the following sequences converge, and which diverge?
Find the limit of each convergent sequence.

(i) $a_n = \left(\frac{4n+1}{4n-1}\right)^n \left(\frac{n^2+8}{n^2+n+1}\right)$ **(ii)** $b_n = \frac{(0.9)^n}{\left(\sqrt[n]{n}+(-1)^n\right)(n!)}$ **(iii)** $c_n = \left(n^{3/\ln n}+4\right)^{1/\ln n}$

Problem 2 (answer on pages 3 and 4 of the booklet.)

(8 pts each) Which of the following series converge, and which diverge?
Find the sum of the series when possible.

(i) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3^{n+1}}+\frac{e^{n+1}}{5^n}\right)$ **(ii)** $\sum_{n=1}^{\infty}\frac{\ln n}{n^{1.5}}$ **(iii)** $\sum_{n=2}^{\infty}\frac{\ln(n!)}{n^3\ln n}$ **(iv)** $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-1-\frac{1}{n}\right)$

Problem 3 (answer on page 5 of the booklet.)

(23 pts) Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty}\frac{(-1)^n\,(x-3)^n}{\left(\sqrt[n]{n}-1\right)n}$$

For what values of x does the series converge absolutely? Conditionally?

Problem 4 (answer on page 6 and last page of the booklet.)

(i) (7 pts) The estimate $\sqrt{1+x} = 1+\frac{x}{2}$ is used when x is small. Estimate the error when $|x| < 0.01$.

(ii) (7 pts) The approximation $e^x = 1+x+\frac{x^2}{2}$ is used when x is small. Use the Remainder estimation theorem to estimate the error when $|x| < 0.1$.

(iii) (7 pts) When $x < 0$, the series for e^x is an alternating series. Use the Alternating Series Estimation theorem to estimate the error that results from replacing e^x by $1+x+\frac{x^2}{2}$ when $-0.1 < x < 0$.

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<u>Problem 1:</u>

i) $a_n = \left(\frac{4n+1}{4n-1}\right)^n \left(\frac{n^2+8}{n^2+n+1}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\cancel{4n}(1+\frac{1/4}{n})}{\cancel{4n}(1-\frac{1/4}{n})}\right)^n \cdot \lim_{n\to\infty}\left(\frac{n^2+8}{n^2+n+1}\right)$$

$$= \frac{e^{1/4}}{e^{-1/4}} \cdot 1$$ By L'Hopitals

$$= e^{1/2}$$

So, $a_n \to e^{1/2}$

ii) $b_n = \dfrac{0.9^n}{\left(\sqrt[n]{n+(-1)^n}\right)(n!)}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n+(-1)^n}} \cdot \lim_{n\to\infty}\frac{0.9^n}{n!}$$

growth of power <<<< growth of factorial

$$= \lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} \cdot 0$$

$$= 1 \cdot 0$$

$$= 0$$

So, $b_n \to 0$

iii) $c_n = \left(n^{3/\ln n} + 4\right)^{1/\ln n}$

Let's check the limit of the inside first:

$$\lim_{n\to\infty} \left(n^{3/\ln n} + 4\right) = \lim_{n\to\infty} e^{\ln\left(n^{3/\ln n}\right)} + 4$$

$$= \lim_{n\to\infty} e^{3\frac{\cancel{\ln n}}{\cancel{\ln n}}} + 4$$

$$= e^3 + 4$$

So, the inside is a constant

$$\Rightarrow \lim_{n\to\infty} c_n = \lim_{n\to\infty} (e^3+4)^{1/\ln n}$$

$$= (e^3+4)^0$$

$$= 1$$

So, $c_n \to 1$.

Problem 2:

i) $\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3^{n+1}}+\frac{e^{n+1}}{5^n}\right]$

$=\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n+e\sum_{n=0}^{\infty}\left(\frac{e}{5}\right)^n$ ... convergent since sum of two geometric series with $|r|=\frac{1}{3}$ & $\frac{e}{5}<1$.

$=\frac{1}{3}\frac{1}{1-\left(-\frac{1}{3}\right)}+e\frac{1}{1-\frac{e}{5}}$

$=\frac{1}{4}+\frac{5e}{5-e}$

ii) $\sum_{n=1}^{\infty}\frac{\ln n}{n^{1.5}}$

Consider $f(x)=\frac{\ln x}{x^{1.5}}$ → $\lim_{x\to\infty}f(x)=0$
→ $f(x)$ is decreasing
→ $f(x)>>0$ for $x>1$.

$\int_1^{\infty}f(x)\,dx=\int_1^{\infty}\frac{\ln x}{x^{1.5}}dx=\int_1^{\infty}\frac{\ln x}{x\sqrt{x}}dx$

$u=\ln x \Rightarrow du=\frac{dx}{x}$

$dv=\frac{dx}{x\sqrt{x}}\Rightarrow v=\int x^{-1.5}dx=\frac{x^{-0.5}}{-0.5}+c=-2x^{-0.5}+c$

$\Rightarrow \int_1^{\infty}f(x)\,dx=\left[-2x^{-0.5}\ln x\right]_1^{\infty}-\int_1^{\infty}-2x^{-0.5}\frac{dx}{x}$

$=\left[-2x^{-0.5}\ln x\right]_1^{\infty}+2\int_1^{\infty}x^{-1.5}dx$

$=\left[-2x^{-0.5}\ln x-4x^{-0.5}\right]_1^{\infty}$

$=0-\left[-\frac{4}{\sqrt{1}}\right]$

$=4 \Rightarrow$ converges by Integral Test!

iii) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n!)}{n^3 \ln n}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n!)}{n^3 \ln n}}{\frac{1}{n^{1.5}}} = \lim_{n\to\infty} \frac{\ln(n!)}{n^{1.5} \ln n} = 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ converges since p-series with $p = 1.5 > 1$.

So, The series converges by LCT.

iv) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - 1 - \frac{1}{n}\right)$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{1/n} = 1 + \frac{1}{n} + \frac{\left(\frac{1}{n}\right)^2}{2!} + \frac{\left(\frac{1}{n}\right)^3}{3!} + \cdots$$

$$e^{1/n} - 1 - \frac{1}{n} = \frac{\left(\frac{1}{n}\right)^2}{2!} + \frac{\left(\frac{1}{n}\right)^3}{3!} + \cdots$$

$$\lim_{n\to\infty} \frac{e^{1/n} - 1 - \frac{1}{n}}{\frac{1}{n^2}} = \frac{1}{2!} + \cancelto{0}{\frac{1}{3!\,n}} + \cdots = \frac{1}{2}$$

& $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since p-series with $p > 1$.

So, converges by LCT.

Problem 3:

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-3)^n}{(\sqrt[n]{n}-1)n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(\sqrt[n+1]{n+1}-1)(n+1)} \cdot \frac{(\sqrt[n]{n}-1)n}{(-1)^n(x-3)^n} \right|$$

$$= |x-3| \cdot \frac{\sqrt[n]{n}-1}{\sqrt[n+1]{n+1}-1} \cdot \frac{n}{n+1} \longrightarrow |x-3|$$

for the series to converge, $|x-3|<1$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

for $x=2$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\sqrt[n]{n}-1)n}$

$$\lim_{n\to\infty} \frac{\frac{1}{(\sqrt[n]{n}-1)n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}-1} = \infty$$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{(\sqrt[n]{n}-1)n}$ diverges by LCT.

for $x=4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\sqrt[n]{n}-1)n}$, $u_n = \dfrac{1}{(\sqrt[n]{n}-1)n}$

$u_n \to 0$, u_n decreasing, $u_n > 0$ for $n \geq 2$.

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\sqrt[n]{n}-1)n}$ converges by AST. (since it diverges absolutely).
(cond.)

So, the series converges absolutely for $2<x<4$, conditionally for $x=4$.

Problem 4:

i) Binomial expansion: $(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} x^n$

$$= 1 + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \cdots$$

$$= 1 + \frac{x}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$$

from the 2nd term & on, $(1+x)^{1/2}$ is alternating

By ASET, $|\text{error}| < |\text{first unused term}|$

$$|\text{error}| < \left|-\frac{(0.01)^2}{8}\right|$$

$$\Rightarrow |\text{error}| < 1.25 \times 10^{-5}$$

iii) for $x<0$, $e^x = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

when replacing e^x by the first three terms:

$|\text{error}| < |\text{first unused term}|$ (By ASET)

for $-0.1 < x < 0$, $|\text{error}| < \left|-\frac{0.1^3}{3!}\right|$

$$|\text{error}| < 1.67 \times 10^{-4}$$