CMPS 251
Assignment 6-Solution

Problem 1

n=1

This corresponds to Euler's method

$$y(t+h) = y(t) + hf\big(t, y(t)\big) = y(t) + h(y^2(t) + y(t)e^t$$

n=2

$$y'' = 2yy' + y'e^t + ye^t$$

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t)$$

n=3:

$$y''' = 2(y')^2 + 2yy'' + y''e^t + y'e^t + y'e^t + ye^t$$

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t)$$

To find y(0.02), let t=0 h=0.02,

$$y'(0) = y^2(0) + y(0)e^0 = 2$$

$$y''(0) = 2y(0)y'(0) + y'(0)e^0 + y(0)e^0 = 7$$

$$y'''(0) = 34$$

Thus,

$$n = 1, \quad y(0.02) = 1 + 0.02(2) = 1.04$$

$$n = 2, y(0.02) = 1 + 0.02(2) + \frac{(0.02)^2}{2}(7) = 1.0414$$

$$n = 3, y(0.02) = 1 + 0.02(2) + \frac{(0.02)^2}{2}(7) + \frac{(0.02)^3}{6}(34) = 1.0414453$$

Problem 2

The implicit solution is found in two steps using Heun's method where we calculate x(t+h) as

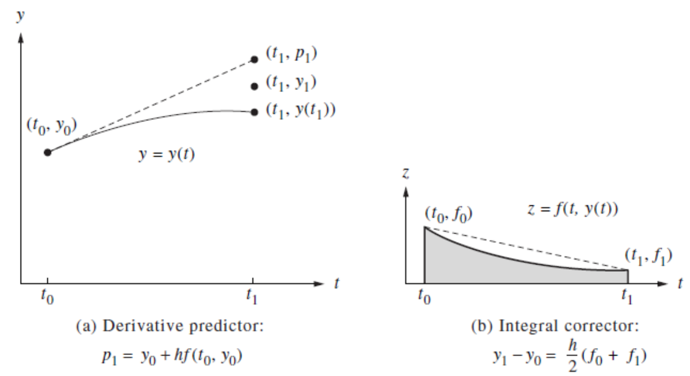
$$x(t+h) = x(t) + \frac{h}{2}\left(f\big(t, x(t)\big) + f\big(t+h, x(t) + hf\big(t, x(t)\big)\big)\right)$$

This is done in two steps

$$p_{k+1} = x_k + hf(t_k, x_k), t_{k+1} = t_k + h$$

$$x_{k+1} = x_k + \frac{h}{2}\left(f(t_k, x_k) + f(t_{k+1}, p_{k+1})\right)$$

This is illustrated in the following figure



Heun's method involves two steps to calculate each solution point and has an error in the order of $O(h^2)$

Problem 3

Using the Runge-Kutta method, we get the following values,

$$x = [2.0000 \quad 2.0564 \quad 2.4147 \quad 3.1902];$$

Using Euler's Method we get

$$x = [2.0000 \quad 2.0000 \quad 2.1715 \quad 2.8033]$$

Actual Solution

$$x = [2.0000 \quad 2.0564 \quad 2.4144 \quad 3.1897];$$

Problem 4

The system can be written as

$$\begin{cases} x' = y \quad x(0) = -1 \\ y' = f(t, x, y) = -2y - x \quad y(0) = 0.2 \end{cases}$$

We use the equations as given in the assignment

$$x_1 = x(0.5) = x(0) + 0.5. y(0) = -1 + 0.5 \cdot 0.2 = -0.9$$

$$y_1 = y(0.5) = y(0) + 0.5 \cdot \big(-2y(0) - x(0)\big) = 0.5$$

i	t_i	x_i	y_i
0	0	-1	0.2
1	0.5	-0.9	0.5
2	1	-0.65	0.45

Now to solve the BVP, we apply this method twice

i	t_i	x_i	y_i
0	0	0	0
1	0.5	0	0
2	1	0	-1

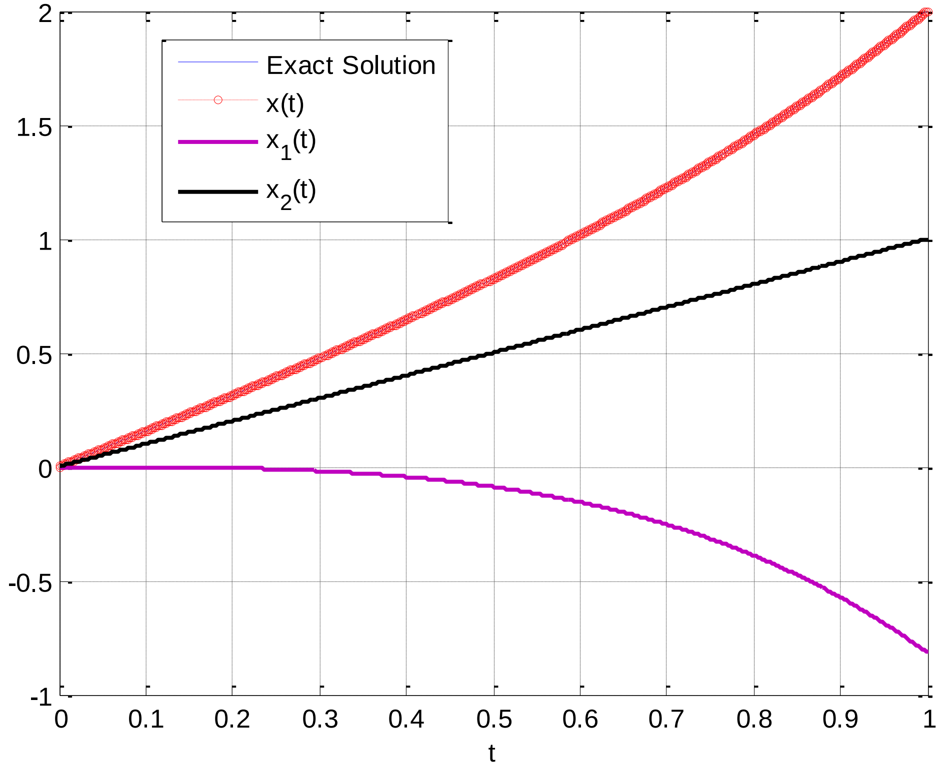
i	t_i	x_i	y_i
0	0	0	1
1	0.5	0.5	1
2	1	1	1

Using a linear shooting method $\lambda = \frac{\beta - x_2(b)}{x_1(b) - x_2(b)} = \frac{2 - x_2(1)}{x_1(1) - x_2(1)} = \frac{2-1}{0-1} = -1$

The solution can be computed as follows

i	$x_1(t_i)$	$x_2(t_i)$	$x(t_i) = \lambda x_1(t_i) + (1-\lambda)x_2(t_i)$
0	0	0	0
1	0	0.5	1
2	0	1	2=β

For h=0.0001, the exact solution coincides with the obtained x(t)



Problem 5

For h=1/2, we have one point to determine

$$\frac{1}{0.25}(x_0 - 2x_1 + x_2) = -4t_1 + 4x_1, x_0 = 0 \quad x_2 = 2$$

$$4(2 - 2x_1) = -2 + 4x_1$$

We get that $x_1 = \frac{5}{6} = 0.83333333$

For h=1/4, we have three points to determine x_1, x_2, x_3

$$\begin{cases} 16(x_0 - 2x_1 + x_2) = -4t_1 + 4x_1 \\ 16(x_1 - 2x_2 + x_3) = -4t_2 + 4x_2 \\ 16(x_2 - 2x_3 + x_4) = -4t_3 + 4x_3 \end{cases}$$

This leads to the following system in three unknowns

$$\begin{cases} -36x_1 + 16x_2 = -4t_1 - 16x_0 = -4(0.25) - 0 = -1 \\ 16x_1 - 36x_2 + 16x_3 = -4t_2 = -2 \\ 16x_2 - 36x_3 = -4t_3 - 16x_4 = -4(0.75) - 32 = -36 \end{cases}$$

Solving the system yields $x_1 = 0.3951, \quad x_2 = 0.8265, \ x_3 = 1.3396$

The extrapolation is the same as the Richardson extrapolation

$$x(0.5) = \frac{4(0.8265) - 0.8333}{3} = 0.8242$$

Problem 6

Part a

Using the continuity of S

$$-a + b - c + d = -1$$

$$a + b + c + d = 1$$

continuity of S'

$$3a - 2b + c = -1$$

$$3a + 2b + c = 1$$

continuity of S''

$$-6a + 2b = 0$$

$$6a + 2b = 0$$

Natural Spline condition (always valid since second derivative is 0)

Last set of equations is satisfied for $a = 0; b = 0$

Replacing in the previous equations shows that it is impossible to find a,b,c,d such that it is a natural cubic spline.

Part b

Using the algorithm presented in the slides we get the following

$$d = [0 \quad 42.5 \quad -30 \quad 20 \quad -25];$$

$$Q = \begin{cases} \dfrac{85(x-14)^2}{32} + 32 & x \in [14; 22] \\ \dfrac{85x}{2} - \dfrac{145(x-22)^2}{32} - 445 & x \in [22; 30] \\ \dfrac{25(x-30)^2}{8} - 30x + 1440 & x \in [30; 38] \\ 20x - \dfrac{45(x-38)^2}{16} - 260 & x \in [38; 46] \end{cases}$$

A speed of 24 mph falls into the second interval so Q(24)=556.875. Q(35)=468.125