CMPS 251
Assignment 5-Solution

Problem 1

This is a least squares problems. We apply the techniques from Section 9.1. Follow example in Slides 21-25 as the derivation is similar except that we have three exponential functions.

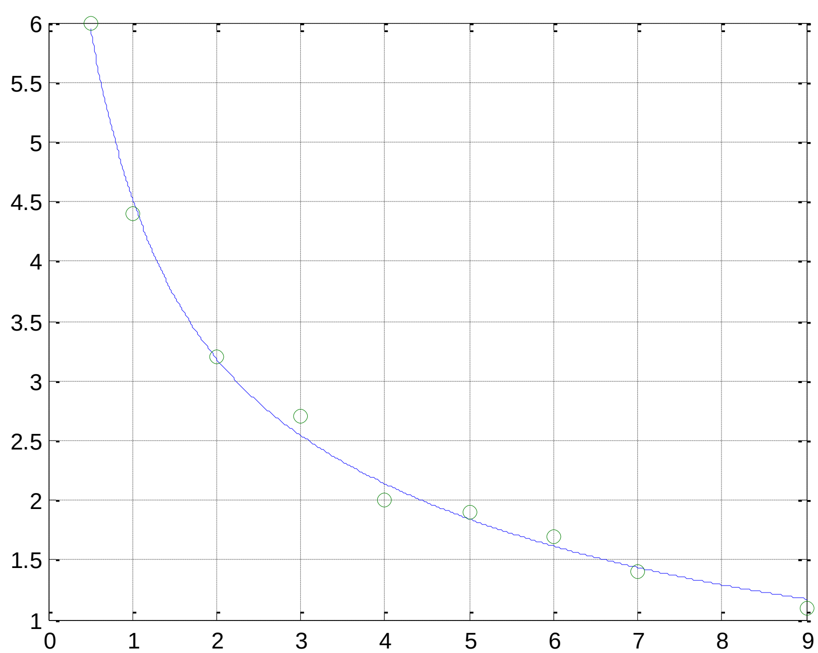
In MATLAB, the following could be done

```
clear;clc
t=[0.5  1   2   3   4   5   6   7   9]';
p=[6 4.4 3.2 2.7 2 1.9 1.7 1.4 1.1]';
A = [exp(-1.5*t),exp(-0.3*t),exp(-0.05*t)];
% get the solution x=[A,B,C]^T
x=(A'*A)\(A'*p);
tt=0.5:0.01:9;
sol=x(1)*exp(-1.5*tt)+x(2)*exp(-0.3*tt)+x(3)*exp(-0.05*tt);
plot(tt, sol);
hold on;
grid on;
scatter(t,p);
```

A=4.1375

B=2.8959

C=1.5349



Problem 2

We have

$$T(h) = I(f; a, a+h) - C_1 h^3 - O(h^\alpha) = I(f; a, a+h) + 2C_2 h^3 - O(h^\alpha)$$
$$M(h) = I(f; a, a+h) - C_2 h^3 - O\left(h^\beta\right)$$

Now

$$T(h) + 2M(h) = 3I(f; a, a+h) - O(h^\gamma)$$
$$I(f; a, a+h) = \frac{T(h) + 2M(h)}{3} + O(h^\gamma)$$

Where γ is strictly greater than 3.

Problem 3

1. $\int_1^2 x \ln x \, dx \qquad n = 4$

   Using the composite trapezoidal rule with $h = \frac{1}{4}$, we get

   $$T(h) = \frac{h}{2}(f(1) + f(2) + 2f(1.25) + 2f(1.5) + 2f(1.75)) = 0.6399$$

   Using the Composite Simpson's rule with $h = \frac{1}{4}$, we get

   $$S(h) = \frac{h}{3}(f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)) = 0.6363$$

2. $\int_0^2 e^{2x} \sin 3x \, dx \quad n = 8$

   Using the composite trapezoidal rule with $h = \frac{1}{4}$, we get

   $$T(h) = \frac{h}{2}(f(0) + f(2) + 2[f(0.25) + f(0.5) + f(0.75) + f(1) + f(1.25) + f(1.5) + f(1.75)]) = -13.576$$

   Using the composite Simpson rule with $h = \frac{1}{4}$, we get

   $$S(h) = \frac{h}{3}(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)) = -14.1833$$

3. Now we need to determine n and h to get an error of 10^{-4}.

   We have $f(x) = e^{2x} \sin 3x$, $f'(x) = e^{2x}(3\cos 3x + 2\sin 3x)$, $f''(x) = e^{2x}(12\cos 3x - 5\sin 3x)$, $f'''(x) = e^{2x}(9\cos 3x - 46\sin 3x)$, $f''''(x) = -e^{2x}(120\cos 3x + 119\sin 3x)$

   Using the Trapezoidal rule, we need $\left|\frac{1}{12}(b-a)h^2 f''(\xi)\right| < 10^{-4}$

   $$\frac{2}{12}h^2 \cdot 705.4 < 10^{-4}$$
   $$h = 9.22 \cdot 10^{-4}$$

$$n > 2168$$

Using the Simpson's Rule we need $\left|\frac{1}{180}(b-a)h^4 f''''(\xi)\right| < 10^{-4}$

$$\frac{1}{180} \cdot 2h^4 \cdot 2845 < 10^{-4}$$

$$h < 0.0422$$

$$n > 47$$

4. $\int_{2.1}^{2.5} \int_{1.2}^{1.4} xy^2 dydx$

   We have n=2 so x is discretized into $x_0 = 2.1, x_1 = 2.3 \; and \; x_2 = 2.5$

   y is discretized into $y_0 = 1.2, y_1 = 1.3, y_2 = 1.4$

   Using a trapezoidal rule, the boundary nodes have a weight of ¼ and the internal nodes a weight of ½

   The integral is given as follows

$$I = \frac{1}{4}\left(\frac{1}{4}(2.1)(1.2)^2 + \frac{1}{2}(2.1)(1.3)^2 + \frac{1}{4}(2.1)(1.4)^2\right) + \frac{1}{2}\left(\frac{1}{4}(2.3)(1.2)^2 + \frac{1}{2}(2.3)(1.3)^2 + \frac{1}{4}(2.3)(1.4)^2\right) + \frac{1}{4}\left(\frac{1}{4}(2.5)(1.2)^2 + \frac{1}{2}(2.5)(1.3)^2 + \frac{1}{4}(2.5)(1.4)^2\right)$$

$$= 3.8985$$

Problem 4

On the interval [-1;1]

We have $I = Af(-1) + Bf\left(-\frac{1}{3}\right) + Cf\left(\frac{1}{3}\right) + Df(1)$

We need to evaluate the integral for the following monomials $1, x, x^2$ and x^3

$$\int_{-1}^{1} dx = 2 = A + B + C + D$$

$$\int_{-1}^{1} x\, dx = 0 = -A - \frac{B}{3} + \frac{C}{3} + D$$

$$\int_{-1}^{1} x^2\, dx = \frac{2}{3} = A + \frac{B}{9} + \frac{C}{9} + D$$

$$\int_{-1}^{1} x^3\, dx = 0 = -A - \frac{B}{27} + \frac{C}{27} + D$$

Solving these equations, we get $A = \frac{1}{4}, B = \frac{3}{4}, C = \frac{3}{4}, D = \frac{1}{4}$

Now using the linear mapping with a weight of $\frac{3h}{2}$

we get $I = \frac{3h}{8}(f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h))$

Now we need to calculate the error

$$Let\ F(x) = \int_a^x f(t)dt$$

$$F(a) = 0$$

$$F(a+3h) = F(a) + 3hF'(a) + \frac{9h^2}{2}F''(a) + \frac{27h^3}{6}F'''(a) + \frac{81h^4}{24}F''''(a) + \frac{243h^5}{120}F^{(5)}(a) + \cdots$$

$$= 3hf + \frac{9h^2}{2}f' + \frac{27h^3}{6}f'' + \frac{81h^4}{24}f''' + \frac{243h^5}{120}f'''' + \cdots$$

$$\frac{3h}{8}\big(f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)\big)$$
$$= \frac{3h}{8}\left(8f + 12hf' + \frac{h^2}{2}(3 + 3.2^2 + 3^2)f'' + \frac{h^3}{6}(3 + 3.2^3 + 3^3)f''' + \frac{h^4}{24}(3 + 3.2^4 + 3^4)\right)f'''' + \cdots$$

$$= 3hf + \frac{9h^2}{2}f' + \frac{27h^3}{6}f'' + \frac{81h^4}{24}f''' + \frac{33h^5}{16}f'''' + \cdots$$

Subtracting the two entities leaves us with a truncation error of $\frac{243h^5}{120}f'''' - \frac{33h^5}{16}f'''' = -\frac{3}{80}h^5 f''''(\xi)$

Problem 5

```
function R=Romberg(f,a,b,n)
h=b-a;
R(1,1)=h/2*(f(a)+f(b));
for i=2:n+1
    h=h/2;
    sum=0;
    for k=1:2:2^(i-1)-1
        sum=sum+f(a+k*h);
    end
        R(i,1)=0.5*R(i-1,1)+sum*h;
        for j=2:i
            R(i,j)=R(i,j-1)+(R(i,j-1)-R(i-1,j-1))/(4^(j-1)-1);
        end
end
```

```
function f=testRom(x)
f=4./(1+x.^2);
```

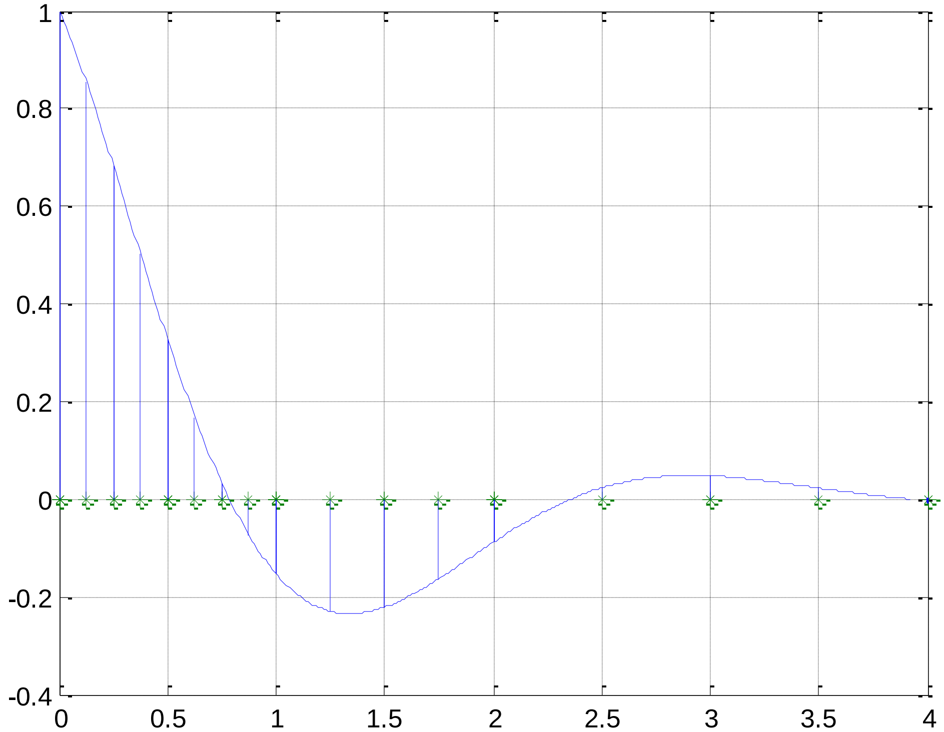
```
R=Romberg(@testRom,0,1,5)
```

Problem 6

```
function [result,x]=Simpson(f,a,b,tol,lev,lev_max)
lev=lev+1;
x=[a,b];
h=b-a;
c=(a+b)/2;
x=[x,c];
one_sim=h*(f(a)+4*f(c)+f(b))/6;
d=(a+c)/2;
e=(c+b)/2;
x=[x,d,e];
two_sim=h*(f(a)+4*f(d)+2*f(c)+4*f(e)+f(b))/12;
if lev>lev_max
    disp('max achieved')
elseif abs(two_sim-one_sim)<15*tol
    result=two_sim+(two_sim-one_sim)/15;
else

    [lef,x1]=Simpson(f,a,c,tol/2,lev,lev_max);
    [right,x2]=Simpson(f,c,b,tol/2,lev,lev_max);
    result=lef+right;
    x=[x,x1,x2];
end
```

```
function y=testfun(x)
y=cos(2.*x)./exp(x);
```


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
4