CMPS 251
Assignment 3- Solution

Problem 1

The code is as follows (note that it makes heavy use of matrix, vector operations and MATLAB inbuilt functions so take your time reading it)

```
function [A,L]=Gauss_procedure(A)
L=1:size(A,1);
s=max(abs(A),[],2); % getting the maximum across each of the rows
for k=1:size(A,1)-1
    A
    [r,index]=max(abs(A(L(k:end),k)./s(L(k:end)))); % find the maximum scaled
pivot element
    L([k index(1)+k-1])=L([index(1)+k-1 k]); % swapping elements in L (the
element is in position index in the ratios vector and k+index in L)
    A(L(k+1:end),k)= A(L(k+1:end),k)/A(L(k),k); % calculating the multipliers
    A(L(k+1:end),k+1:end)=A(L(k+1:end),k+1:end)-
repmat(A(L(k),k+1:end),[size(A,1)-k 1]).*repmat(A(L(k+1:end),k), [1
size(A,1)-k]);
end


function x=Solve_procedure(A,L,b)
x=zeros(size(A,1),1);
for k=1:size(A,1)-1
    b(L(k+1:end))=b(L(k+1:end))-A(L(k+1:end),k)*b(L(k));
end
x(end)=b(L(end))/A(L(end),end);
for i=length(L)-1:-1:1
    x(i)=(b(L(i))-A(L(i),i+1:end)*x(i+1:end))/A(L(i),i);
    x(i)
end
```

Example to run the code

```
clear;clc
A=[3 -13 9 3;-6 4 1 -18;6 -2 2 4;12 -8 6 10];
b=[-19;-34;16;26];
tic
[A,L]=Gauss_procedure(A);
x=Solve_procedure(A,L,b);
time=toc
```

And the output is as follows

```
A =
```

```
   3   -13     9     3
  -6     4     1   -18
   6    -2     2     4
  12    -8     6    10


A =

    0.5000  -12.0000    8.0000    1.0000
   -1.0000    2.0000    3.0000  -14.0000
    6.0000   -2.0000    2.0000    4.0000
    2.0000   -4.0000    2.0000    2.0000


A =

    0.5000  -12.0000    8.0000    1.0000
   -1.0000   -0.1667    4.3333  -13.8333
    6.0000   -2.0000    2.0000    4.0000
    2.0000    0.3333   -0.6667    1.6667


time =

    0.0066


x =

    3.0000
    1.0000
   -2.0000
    1.0000
```

<u>Problem 2</u>

Each of the points satisfies the equation of the circle. For a point with coordinates (α,β), the following can be observed

$$(\alpha - a)^2 + (\beta - b)^2 = r^2$$

Expanding this expression, we get

$$2a\alpha + 2b\beta + (r^2 - b^2 - a^2) = \alpha^2 + \beta^2$$

Let $c = r^2 - b^2 - a^2$, we get the system of the following linear equations

$$2\alpha \cdot a + 2\beta \cdot b + c = \alpha^2 + \beta^2$$

For the points (-1; 3.2),(-8; 4), and (-6.5;-9.3), we get the system of linear equation with the following matrices

$$A = \begin{bmatrix} -2 & 6.4 & 1 \\ -16 & 8 & 1 \\ -13 & -18.6 & 1 \end{bmatrix}, b = \begin{bmatrix} 11.24 \\ 80 \\ 128.74 \end{bmatrix}$$

Applying the naïve Gaussian elimination results in the following matrices

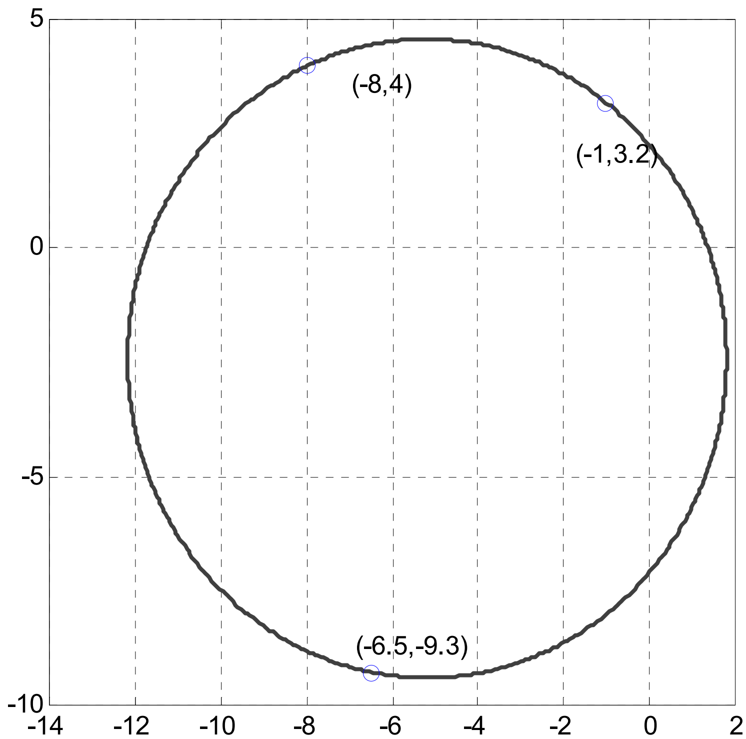
$$A = \begin{bmatrix} -2 & 6.4 & 1 \\ 0 & -43.2 & -7 \\ 0 & 0 & 4.2546 \end{bmatrix}, b = \begin{bmatrix} 11.24 \\ -9.92 \\ 69.4637 \end{bmatrix}$$

And the resulting solution $[a\ b\ c]^T = [-5.1875 \quad -2.4159 \quad 16.3266]^T$

Finally the radius is $r = \sqrt{c + a^2 + b^2} = \sqrt{49.0736} = 7.0053$

And the equation of the circle is

$$(x + 5.1875)^2 + (y + 2.4159)^2 = 49.0736$$



Matlab's A\b yields the following

```
>> A\b

ans =

  -5.1875
  -2.4159
  16.3266
```

Problem 3

The scaled partial pivoting routine from problem 1 can be used to solve the system of equations giving the following solution x=[4 4]^T

Now, by perturbating the matrix A and the vector b as required in the assignment, the new solution becomes $[3.98\ 4.04]^T$. By trying for instance to perturbate A without perturbating b, the solution becomes $[2.02\ 6.91]^T$

We can see that the system of equations is sensitive to the slightest changes in the coefficients. We suspect that the matrix is ill-conditioned.

To verify this we calculate the condition number

$$A = \begin{bmatrix} 34.9 & 23.6 \\ 22.9 & 15.6 \end{bmatrix} \qquad \|A\|_\infty = \max\{58.5, 38.5\} = 58.5$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 15.6 & -23.6 \\ -22.9 & 34.9 \end{bmatrix} \|A^{-1}\|_\infty = \max\{9.8, 14.45\} = 14.45$$

The condition number is then

$$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty = 845.325$$

Thus when solving Ax=b, one is expected to lose $\log_{10} \kappa(A) \sim 3$ digits

Problem 4

For the matrix A to be strictly diagonally dominant, it should satisfy the following condition

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \ \forall i$$

For the given matrix we have the following equations

$$4 > \alpha + 1 \Rightarrow \alpha < 3$$

$$5 > 2\beta + 4 \Rightarrow \beta < \frac{1}{2}$$

$$\alpha > 2 + \beta$$

Part 2

The two systems can be solved using the scaled partial pivoting approach developed in Problem 1. For the first system and using single precision by applying the Matlab command $\mathrm{single}()$.

The code used the find all the parameters in this problem is as follows

```
clear;clc
A=single([1 1/2 1/3 1/4 1/5;1/2 1/3 1/4 1/5 1/6;1/3 1/4 1/5 1/6 1/7;1/4 1/5
1/6 1/7 1/8;1/5 1/6 1/7 1/8 1/9]);
b=[1;0;0;0;0];
[A2,L]=Gauss_procedure(A);
x1=Solve_procedure(A2,L,b)
r1=norm(A*x1-b)
x1tr=[25;-300; 1050;-1400; 630];
e1=norm(x1tr-x1)
Ar=single([1.0 0.5 0.333333 0.25 0.2;0.5 0.333333 0.25 0.2 0.166667;0.333333
0.25 0.2 0.166667 0.142857;0.25 0.2 0.166667 0.142857 0.125;0.2 0.166667
0.142857 0.125 0.111111]);
[A3,L]=Gauss_procedure(Ar);
x2=Solve_procedure(A3,L,b)
```

```
r2=norm(Ar*x2-b)
x2tr=[26.9314;-336.018; 1205.11;-1634.03;744.411];
e2=norm(x2tr-x2)
```

The resulting solution, residual and errors are as follows

```
x1 =

   1.0e+03 *

   0.025002395629883
  -0.300038787841797
   1.050148559570312
  -1.400204223632813
   0.630092590332031


r1 =

   2.6428997e-05


e1 =

   0.271772779045329


x2 =

   1.0e+03 *

   0.026937973022461
  -0.336147918701172
   1.205691040039063
  -1.634926513671875
   0.744854919433594


r2 =

   3.5785033e-05

e2 =

   1.164186766766151
```

We can note that by approximating the coefficients of A to at most six significant digits, the exact solution changed considerably. Moreover, when comparing the l_2-norm of the residual and error vector, we can clearly see that both increased. Also, if we compare the condition number, we can see that the matrix with approximated coefficients has a higher condition number.

Problem 5

Part 1

The matrix A is as follows

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

Using Doolittle's algorithm

$$l_{11} = 1, l_{22} = 1, l_{33} = 1$$
$$u_{11} = 2, u_{12} = -1, u_{13} = 1$$
$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{2}; l_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2}$$
$$u_{22} = a_{22} - l_{21}u_{12} = \frac{9}{2}$$
$$u_{23} = a_{23} - l_{21}u_{13} = \frac{15}{2}$$
$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = 1$$
$$u_{33} = (a_{33} - (l_{31}u_{13} + l_{32}u_{23}) = -4$$

At the end we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & -4 \end{bmatrix}$$

Now, we first solve $Lz = b$

$$z_1 = b_1 = -1, \frac{3}{2}z_1 + z_2 = 0 \Rightarrow z_2 = \frac{3}{2}, \frac{3}{2}z_1 + z_2 + z_3 = 4 \Rightarrow z_3 = 4$$

Finally, we solve $Ux = z$

$$-4x_3 = z_3 \Rightarrow x_3 = -1, \frac{9}{2}x_2 + \frac{15}{2}x_3 = \frac{3}{2} \Rightarrow x_2 = 2, 2x_1 - x_2 + x_3 = 1 \Rightarrow x_1 = 1$$

The solution is therefore $[1 \quad 2 \quad -1]^T$

Part 2

For the LDL^T factorization, we use the algorithm presented in the slides

We get the following updates

For j=1

$$l_{11} = 1,\ \ d_1 = a_{11} = 1, l_{21} = \frac{a_{21}}{d_1} = 2,\ \ l_{31} = \frac{a_{31}}{d_1} = -1$$

For j=2

$$l_{22} = 1, d_2 = a_{22} - d_1 l_{21}^2 = -1, \qquad l_{32} = \frac{a_{32} - l_{31}d_1 l_{21}}{d_2} = 2,\ \ l_{42} = \frac{a_{42} - l_{41}d_1 l_{21}}{d_2} = -1$$

For j=3

$$l_{33} = 1, d_3 = a_{33} - (d_1 l_{31}^2 + d_2 l_{32}^2) = 2, l_{43} = \frac{a_{43} - (l_{41} d_1 l_{31} + l_{42} d_2 l_{32})}{d_3} = 1$$

for j=4

$$l_{44} = 1, d_4 = a_{44} - (d_1 l_{41}^2 + d_2 l_{42}^2 + d_3 l_{43}^2) = -2$$

Finally the L and D matrices are as follows

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

For the Cholesky factorization, again we use the algorithm in the slides

For k=1

$$l_{11} = \sqrt{a_{11}} = \sqrt{6}, \qquad l_{21} = \frac{a_{21}}{l_{11}} = \frac{15}{\sqrt{6}}, l_{31} = \frac{a_{31}}{l_{11}} = \frac{55}{\sqrt{6}}$$

For k=2

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{\frac{105}{6}}, l_{32} = \frac{a_{32} - l_{31} l_{21}}{l_{22}} = \frac{525\sqrt{630}}{630}$$

For k=3

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{\frac{34860}{630}}$$

Taking approximations, the resulting matrix L is as follows

$$L = \begin{bmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 7.4386 \end{bmatrix}$$

Part 3

For this matrix, we need to permute rows 1 and 2 to be able to do Gaussian elimination. The permutation matrix is therefore as follows

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$