Math 201 Quiz 1 (Time: 65 minutes) (Fall 2012) *N. Nahlus*

Name: ……………………………………………… (VERY CLEARLY)

I.D ....

Circle your Section number (- 3 points if incorrect)

Sec 8 (12:30 T) — Sec 9 (2:00 T) — Sec 10 (11:00 T) — Sec 11 (5:00 T) ---

Sec 12 (9:00 M) — Sec 13 (3:30 R) — Sec 14 (10:00 M) — Sec 15 (5:00 R) ---

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Investigate = Investigate (with justification) convergence or divergence of the following series.

In LCT problems: IF you know the answer of $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ simply write the answer **without proof.**

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1. (24%) (a) Investigate $\sum_{n=1}^{\infty} \left(\frac{5n+1}{5n+7}\right)^n$

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Total over 100	

(b) Investiagte $\sum_{n=1}^{\infty} a_n$ given that $\left|\frac{a_{n+1}}{a_n}\right| = \frac{4n^2}{(3n+2)(n+1)} - \frac{\sin(n!)}{\ln n}$

(c) Investiagte $\sum_{n=1}^{\infty} a_n$ given that $\left|\frac{a_{n+1}}{a_n}\right| = 1 + \frac{n!}{n^n}$

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d) Investigate $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n} - 1}{n^2}$ (Hint: Write $\sqrt[n]{n}$ as exponential. Then use exp. Series.)

2a) (9%) Find $\lim\limits_{n\to\infty} \dfrac{n^3}{\sqrt[n]{(3n)!}}$

<u>Hint</u>: Use $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ (if this last exists) . Moreover, $\dfrac{n^3}{\sqrt[n]{(3n)!}} = \sqrt[n]{\dfrac{\cdots\cdot}{\cdots\cdot\cdot}}$

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2b) (10%) Find the open interval of convergence $\sum\limits_{n=2}^{\infty} (-1)^n \dfrac{(2x-3)^n}{7^n \, n \ln n}$

Do **<u>not</u>** check end-points

3) (19%) a) (10%) Find the Maclaurin series of $f(x)=\dfrac{e^{7x^3}-1}{x}$ to deduce $f^{(149)}(0)$

3b) (9%) Find the exact value of $\sum_{n=0}^{\infty}(-1)^n\dfrac{n(n+3)}{2^n}$

(<u>Hint</u>: Differentiate the series $\dfrac{1}{1+x}$, then multiply by x^4, then differentiate again.)

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4) (19%) **<u>Suppose</u>** $\sum_{n=1}^{\infty} a_n$ **converges,** prove or disprove that $\sum_{n=1}^{\infty} \ln(2+\sin(a_n))$ converges

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4b) **<u>Suppose</u>** $\sum_{n=1}^{\infty} a_n$ **converges,** prove or disprove that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{5+\sin(a_n)}{4}\right)^n$ converges

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4c) If $a_1 = 1$ and $\frac{a_{n+1}}{a_n} = \frac{n}{n+1}$ for all n, investigate convergence/divergence of $\sum_{n=1}^{\infty} a_n$

(<u>Hint</u>: First, find $a_1, a_2, a_3, a_4, \ldots\ldots.. a_n$)

5) (19%) a) (8%) **Find** $\lim_{n\to\infty} \dfrac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots\dots+\frac{1}{\sqrt{n}}}{\sqrt{5n}}$

5b) (5%) Use the Maclaurin series of e^x to prove $\sqrt[n]{n!} > \dfrac{n}{e}$ for large n.

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MATH 201 / Nazih Nahlus / Quiz I / FALL 2012

1) a) $\sum_{n=1}^{\infty} \left(\frac{5n+1}{5n+7}\right)^n$

$$\lim_{n\to\infty} \left(\frac{5n+1}{5n+7}\right)^n = \lim_{n\to\infty} \left(\frac{\cancel{5n}\left(1+\frac{1/5}{n}\right)}{\cancel{5n}\left(1+\frac{7/5}{n}\right)}\right)^n = \frac{e^{1/5}}{e^{7/5}} = e^{-6/5} \neq 0$$

So, It diverges by n^{th} term test.

b) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4n^2}{(3n+2)(n+1)} + \lim_{n\to\infty} \left(-\frac{\sin(n!)}{\ln n}\right)$$

$$= \frac{4}{3} - 0 = \frac{4}{3} > 1$$

$$-1 \leq \sin(n!) \leq 1$$

$$-\frac{1}{\ln n} \leq -\frac{\sin(n!)}{\ln n} \leq \frac{1}{\ln n}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$

So, $-\frac{\sin(n!)}{\ln n} \to 0$ by Sandwich theorem

So, It diverges by Ratio Test.

c) $\left|\frac{a_{n+1}}{a_n}\right| = 1 + \frac{n!}{n^n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow |a_n|$ is increasing

So, $\lim_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim_{n\to\infty} a_n \neq 0$

So, It diverges by n^{th} term test.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}-1}{n^2}$

$$\sqrt[n]{n} = e^{\ln(\sqrt[n]{n})} = e^{\frac{\ln n}{n}}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{\frac{\ln n}{n}} = 1 + \frac{\ln n}{n} + \frac{\frac{\ln^2 n}{n^2}}{2!} + \frac{\frac{\ln^3 n}{n^3}}{3!} + \cdots$$

$$e^{\ln n/n} - 1 = \frac{\ln n}{n} + \frac{\frac{\ln^2 n}{n^2}}{2!} + \frac{\frac{\ln^3 n}{n^3}}{3!} + \cdots$$

$$\frac{e^{\ln n/n} - 1}{n^2} = \frac{\ln n}{n^3} + \frac{\frac{\ln^2 n}{n^4}}{2!} + \frac{\frac{\ln^3 n}{n^5}}{3!} + \cdots$$

$$\lim_{n\to\infty} \frac{\frac{e^{\ln n/n} - 1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n} + \frac{\frac{\ln^2 n}{n^2}}{2!} + \frac{\frac{\ln^3 n}{n^3}}{3!} + \cdots = 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since p-series with $p = 2 > 1$

So, It converges By LCT

2) a) $\lim\limits_{n\to\infty} \dfrac{n^3}{\sqrt[n]{(3n)!}} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^{3n}}{(3n)!}}$, $|a_n| = \dfrac{n^{3n}}{(3n)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{3n+3}}{(3n+3)!} \cdot \frac{(3n)!}{n^{3n}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{3n} \cdot (n+1)^3 \cdot \frac{\cancel{(3n)!}}{\cancel{(3n)!}(3n+1)(3n+2)(3n+3)}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{3n} \cdot \frac{(n+1)^{\cancel{3}2}}{3(3n+1)(3n+2)\cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^3 \cdot \frac{1}{3} \cdot \frac{(n+1)^2}{(3n+1)(3n+2)}$$

$$= e^3 \cdot \frac{1}{3} \cdot \frac{1}{9}$$

$$= \frac{e^3}{27}$$

$\Rightarrow \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists

$\Rightarrow \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$

So, $\lim\limits_{n\to\infty} \dfrac{n^3}{\sqrt[n]{(3n)!}} = \dfrac{e^3}{27}$

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b) $\sum_{n=2}^{\infty} (-1)^n \frac{(2x-3)^n}{7^n n \ln n}$

Using ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-3|^{n+1}}{7^{n+1}(n+1)\ln(n+1)} \cdot \frac{7^n n \ln n}{|2x-3|^n}$$

$$= \frac{1}{7} \frac{n}{n+1} \frac{\ln n}{\ln(n+1)} |2x-3| \longrightarrow \frac{1}{7}|2x-3|$$

$$\frac{1}{7}|2x-3| < 1$$

$$-7 < 2x-3 < 7$$

$$-4 < 2x < 10$$

So, $-2 < x < 5$ ... the open interval of convergence

3) a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{7x^3} = 1 + 7x^3 + \frac{49x^6}{2!} + \frac{343x^9}{3!} + \cdots$$

$$e^{7x^3} - 1 = 7x^3 + \frac{49x^6}{2!} + \frac{343x^9}{3!} + \cdots$$

$$\frac{e^{7x^3}-1}{x} = f(x) = 7x^2 + \frac{49x^5}{2!} + \frac{343x^8}{3!} + \cdots$$

So, $f(x) = \sum_{n=1}^{\infty} \frac{7^n x^{3n-1}}{n!}$

The regular Maclaurin series form is $\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

To find $f^{(149)}(0)$, take into consideration: - the 149th term of the Maclaurin series form

- the 50th term of $f(x)$ (since $3(50)-1=149$)

$$\Rightarrow \frac{f^{(149)}(0)\,\cancel{x^{149}}}{149!} = \frac{7^{50}\,\cancel{x^{149}}}{50!}$$

So, $f^{(149)}(0) = 7^{50}\,\dfrac{149!}{50!}$

b) $$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=0}^{\infty}(-1)^n n x^{n-1} = -\frac{1}{(1+x)^2}$$

$$\sum_{n=0}^{\infty}(-1)^n n x^{n+3} = -\frac{x^4}{(1+x)^2}$$

$$\sum_{n=0}^{\infty}(-1)^n n(n+3)x^{n+2} = -\frac{2x^3(x+2)}{(1+x)^3}$$

$$\sum_{n=0}^{\infty}(-1)^n n(n+3)x^{n} = -\frac{2x(x+2)}{(1+x)^3}$$

for $x=\frac{1}{2}$: $$\sum_{n=0}^{\infty}\frac{(-1)^n n(n+3)}{2^n} = \frac{-2\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)}{\left(1+\frac{1}{2}\right)^3} = -\frac{20}{27}$$

So, $$\sum_{n=0}^{\infty}\frac{(-1)^n n(n+3)}{2^n} = \frac{-20}{27}$$

4) a) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

So, $\lim_{n\to\infty} \sin(a_n) = 0$

$$\lim_{n\to\infty} (\sin(a_n) + 2) = 2$$

$$\lim_{n\to\infty} \ln(\sin(a_n) + 2) = \ln 2 \neq 0$$

So, $\sum_{n=1}^{\infty} \ln(2+\sin(a_n))$ diverges by n^{th} term test.

b) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

So, $\lim_{n\to\infty} \left(\frac{5+\sin(a_n)}{4}\right)^n = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty \neq 0$

This does not satisfy a condition of Leibniz's test for alternating series.

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \left(\frac{5+\sin(a_n)}{4}\right)^n$ diverges by AST.

c) $\frac{a_{n+1}}{a_n} = \frac{n}{n+1}$ & $a_1 = 1$

$\Rightarrow \frac{a_2}{a_1} = \frac{1}{2}$, $a_2 = \frac{1}{2}$

$\frac{a_3}{a_2} = \frac{2}{3}$, $a_3 = \frac{1}{3}$

$\frac{a_4}{a_3} = \frac{3}{4}$, $a_4 = \frac{1}{4}$

$\vdots$

So, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series (p-series with p=1)

5) a) $$\frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}}{\sqrt{5n}} = \frac{S_n}{\sqrt{5n}}$$

We use: $$\int_1^{n+1} x^{-1/2}\,dx < S_n < 1+\int_1^{n} x^{-1/2}\,dx$$

$$\left[2x^{1/2}\right]_1^{n+1} < S_n < 1+\left[2x^{1/2}\right]_1^{n}$$

$$2\left((n+1)^{1/2}-1\right) < S_n < 1+2\left(n^{1/2}-1\right)$$

$$\frac{2\left((n+1)^{1/2}-1\right)}{\sqrt{5n}} < \frac{S_n}{\sqrt{5n}} < \frac{2n^{1/2}-1}{\sqrt{5n}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{2}{\sqrt{5}} \qquad\qquad\qquad \frac{2}{\sqrt{5}}$$

$\Rightarrow \dfrac{S_n}{\sqrt{5n}} \to \dfrac{2}{\sqrt{5}}$ By sandwich theorem

So, $$\lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}}{\sqrt{5n}} = \frac{2}{\sqrt{5}}$$