**Problem 1 (10 Points)**

Consider the following table of data values for a certain function *f*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | -1 | 0 | 1 | 2 |
| *f(x)* | -1 | 0 | 2 | 5 |

a. (5) Give the interpolating polynomial for the given data in Lagrange form

b. (5) Give the interpolating polynomial for the given data in Newton’s form

**Problem 2(15 Points)**

Again, consider the following table of data values for a certain function *f*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | -1 | 0 | 1 | 2 |
| *f(x)* | -1 | 0 | 2 | 5 |

1. (7) What are the linear hat functions associated with this partition of [-1, 2]?

b. (8) Find the linear spline function that interpolates this data.

**Problem 3 (10 Points)**

Is there a choice for the coefficients *a, b, c, and d* such that the following function is a natural cubic spline:

 *x*  + 1 , -2*x*  *-*1

 *f* ( *x* ) = *ax3* + *bx2* + *c x + d* -1*x* 1

 *x -* 1 1 *x* 2

Justify your answer.

**Problem 4 (15 Points)**

1. (8) Show how to derive the *centered difference formula* for approximating *f’* :

 *f’(a) =*  [ *f(a+h) - f(a-h)*] */ 2h* + O (*h* )

b. (7) Give a precise term that represents O (*h* ) in the above, and state the continuity conditions on *f* that are needed to derive this estimate.

**Problem 5 ( 20 Points)**

Consider the Trapezoid Rule, *T(h),* gives an approximation for the definite integral I(*f* ; *a, a+h*) of a function *f*  on the interval (*a, a+h*) as follows:

I(*f* ; *a, a+h*) = *T(h)* + *ET* (*h*) , where *T(h) = h* [ *f*(*a*) + *f*(*a + h*) ] / 2

1. (5) Use the formula for *T(h*) to derive the “composite trapezoid rule” for approximating the integral I( *f*; *a*, *b*) of *f*  on the interval (*a, b*), using a uniform partition of *n* equal subintervals each of length *h.*
2. (5) Suppose that you know that *ET* (*h* ) = - *h*3  *f”*(/12 for some in (*a, a+h*). Show that the global error in the composite trapezoid rule of (a) above is given by:

 *ECT* (*h* ) = *-h*2 (b-a) *f”* (/12 for some  in (*a, b*).

1. (10) Using (c) or otherwise, derive how large *n* must be if the method in (a) is to estimate the integral of the function *f*(*x*) = ex on the interval (0, 1) with an error not exceeding 104

P**roblem 6 (15 Points)**

Again this problem refers to the composite trapezoid rule that you have derived in the previous problem. Suppose that you know that

 *ET* (*h* ) = *h*2 ( *f* ’(*a+h*) – *f* ’(*a*) ) / 12 + *f*(4) ( ) *h*5 / 720 + higher order terms

1. (10) Refine the formula that you have derived for the composite trapezoid rule to get an O(*h*4) method, assuming that *f* ’(*a*) and *f* ’(*b*) are given.
2. (5) Using (a) or otherwise, show that the composite trapezoid rule is an O(*h*4) method for periodic functions whose first derivative has equal values at *a* and at *b.*

**Problem 7 (15 Points) Extrapolation!!**

The Trapezoid Rule, *T(h),* and the Mid-point Rule, *M(h),* give two approximations for the definite integral I(*f* ; *a, a+h*) of a function *f*  on the interval (*a, a+h*) as follows:

 I(*f* ; *a, a+h*) = *T(h)* + C1 *h*3 + O (*h* ), with *T(h) = h* [ *f*(*a*) + *f*(*a + h*) ] / 2

 I(*f* ; *a, a+h*) = *M(h)* + C2 *h*3 + O (*h* ), with *M(h) = h* *f*(*a + h/2* )

where C1 = - 2 C2 , and and  > 3, if *f*  is sufficiently continuous.

a. (10) Taking a linear combination of *T(h)*  and *M(h),* derive a formula for approximating I(*f*; *a, a+h*), with an error of order ***higher*** than 3.

b. (5) Give an interpretation of the formula that you get in (a)