CMPS 251: *Numerical Computing* February 2nd, 2006

Final Exam

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Duration: 120 minutes

Notes:

1. You are allowed to use no more than two $A4$ pages of notes (front and back).
2. This exam consists of seven questions. Make sure you have all of them.

Question 1 (8pts)

(a) Let f be an infinitely differentiable function in a closed interval $I = [\alpha, \beta]$. Let h be any value such that $x + h \in I$ and consider $a, b \in \mathbb{R}$. Use Taylor series to show that f' can be approximated using the formula

$$f'(x) \approx \frac{1}{(b-a)h} \left[bf(x+h) - f(x+ah) - (b-1)f(x) \right] \tag{1}$$

(5pts)

(b) Establish the error term for Equation (1) above. (3pts)

Solution:

Solution cont'd:

Question 2 (9pts)

Consider the following system of equations:

$$\begin{cases} x_1 + \alpha x_2 - 3x_3 &= 7 \\ 12x_1 + 12\alpha x_2 + x_3 &= 121 \\ \alpha x_1 + 4x_2 + 2x_3 &= 6 \end{cases}$$

where $\alpha \in \mathbb{R}$. For which values of α does the above system possess

1. no solution? (3pts)
2. infinitely many solutions? (3pts)
3. a unique solution (to be determined if it exists)? (3pts)

Solution:

Solution cont'd:

Question 3 (21pts)

A *symmetric pentadiagonal system* is a pentadiagonal system in which the subdiagonals are the same as the superdiagonals.

(a) Develop a pseudo-code to solve a symmetric pentadiagonal system using scaled partial pivoting. (7pts)

(b) Determine the operational count (run-time in terms of the input size) of your pseudo-code. (7pts)

(c) Is this more efficient than the pseudo-code for solving general pentadiagonal systems? Justify your answer in terms of operational and/or spatial costs, whenever applicable. (7pts)

Solution:

Solution cont'd:

Question 4 **(18pts)**

(a) What is the minimum number of sub-intervals required to insure that the error in approximating $\int_1^2 \ln x^3 dx$ using the trapezoid rule is less than 5×10^{-5}? (5pts)

(b) Consider the integral $\int_0^1 e^x dx$.

- Use the trapezoid rule to find the value of the integral with 2, 4, and 8 sub-intervals. (5pts)
- Use Romberg integration to obtain the best approximation of the integral from the results above. (5pts)
- What is the order of the approximation of the Romberg rule above? (i.e., what is the value of k if the error is written as $O(h^k)$ where h is the interval size). (3pts)

Solution:

Solution cont'd:

Question 5 (20pts)

(a) Consider the trapezoid rule for the interval $[0, 1]$ using $n + 1$ equally spaced points. Explain why the composite trapezoid rule can be expressed as

$$\int_0^1 f(x)dx \approx \frac{h}{2}\left[f(0) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right]$$

where h is the step size. (5pts)

(b) Rewriting the above expression as

$$\int_0^1 f(x)dx \approx \sum_{i=0}^{n} A_i f\left(\frac{i}{n}\right)$$

how would the A_i's be defined? (5pts)

(c) Consider the composite trapezoid rule for double integration of bivariate functions over the unit square $[0, 1] \times [0, 1]$:

$$\int_0^1 \int_0^1 f(x, y)dxdy \approx \sum_{i=0}^{n}\sum_{j=0}^{n} A_i A_j f\left(\frac{i}{n}, \frac{j}{n}\right)$$

Use the above expression to develop a pseudo-code approximating

$$\int\int e^{-(x^2+y^2)} dxdy$$

over the unit square. (10pts)

Solution:

Solution cont'd:

Question 6 (12pts)

Consider an iterative solution of the following 2×2 set of linear equations:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

(a) Determine the Jacobi iteration matrix. (4pts)

(b) Will Jacobi converge? Why or why not? (4pts)

(c) Perform one iteration of Jacobi with the starting value $x^0 = (0, 0)$. (4pts)

Solution:

Solution cont'd:

Question 7 (12pts)

(a) Determine the values of a and b that make the function

$$f(x) = \begin{cases} x^2 & x \in [0,1] \\ \frac{1}{2}(x-1)^2 + a(x-1) + b & x \in [1,3] \end{cases}$$

a quadratic spline. (4pts)

(b) In class, we considered the problem of interpolating $n+1$ data points with co-ordinates (x_i, y_i), $i = 1, \cdots, n+1$, using a quadratic spline $Q(x)$. The continuity and smoothness conditions at x allowed us to express the n spline segements as:

$$q_i(x) = \frac{z_{i+1} - z_i}{2(x_{i+1} - x_i)}(x - x_i)^2 + z_i(x - x_i) + y_i, \quad x \in [x_i, x_{i+1}]$$

where $z_{i+1} = 2\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - z_i$, $z_i = Q'(x_i)$. In order to determine the parameters z_i uniquely we needed one additional condition. In class, we imposed $z_1 = 0$ and used it to compute the remaining z_i's.

- Explain how you would compute the unknown coefficients z_i if the extra condition becomes alternatively $z_1 = z_2$. (4pts)
- What is the geometric interpretation of this condition? (4pts)

Solution:

Solution cont'd: