AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 251
FALL 2000
FINAL EXAM
Wednesday Jan. 31st, 2001; 2:00-4:00 pm
Closed Book, Two hours
Calculators are allowed

1. Consider a set of data $\{(x_i, y_i) \,|\, i = 1, ..., n\}$, with $x_i \neq x_j$, $i \neq j$.

   - State and prove the theorem on the existence and uniqueness of the Lagrange interpolation polynomial $p_{n-1}(x)$

   $$p_{n-1}(x_i) = y_i, \; i = 1, ..., n.$$

   (10 points)

   - Write the MATLAB `function Y=DivDiff(x,y)` that constructs the divided difference table associated with the data

   $$\{(x_i, y_i) | \, i = 1, .., n\}.$$

   The input data are the column vectors of length n `x, y`. The output $n \times n$ matrix `Y` is lower-triangular, such that the first column of `Y` is the vector column `y`. (10 points)

2. Consider a uniform partition of the interval $[a, b]$ into n equidistant intervals of length $h = \frac{b-a}{n}$:

   $$a = x_1 < x_2 = a+h < ... < x_i = a+(i-1)h < ... < x_{n+1} = a+nh = b.$$

   Let $f(x)$ be a function defined on $[a, b]$, with f, f', ..., $f^{(4)}$ continuous on $[a, b]$. ($f \in C^4([a, b])$). We adopt the following notations : $\{f_i = f(x_i), i = 1, ..., n+1\}$ and $f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}})$ where $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$, is the middle point of the interval $[x_i, x_{i+1}]$, $i = 1, ..., n$. Let $I = \int_a^b f(x)dx$.

   - Derive the composite trapezoïdal rule $T(h)$ to approximate I. Give (without proof) the expression of $I - T(h)$ (5 points).

- Derive the composite Simpson's rule $S(h)$ to approximate I. Give (without proof) the expression of $I - S(h)$. (5 points)
- Show that

$$3S(h) = T(h) + 2h \sum_{i=1}^{n} f_{i+\frac{1}{2}}.$$

(5 points)

- By writing $T(h) = \frac{h}{2}(f_1 + f_{n+1}) + \sum_{i=2}^{n} f_i$. show that :

$$2T(\frac{h}{2}) = T(h) + h \sum_{i=1}^{n} f_{i+\frac{1}{2}}.$$

(5 points)

- Derive the relationships between $T(h)$, $T(\frac{h}{2})$ and $S(h)$. (5 points)
- Consider the following table of data, associated with $f(x)$ on the interval $[0, 2]$.

x	$f(x)$
0.00	1.000
0.25	1.284
0.50	1.649
0.75	2.117
1.00	2.718
1.25	3.490
1.50	4.482
1.75	5.755
2.00	7.389

Find $T(1)$, $T(0.5)$, $T(0.25)$, $S(0.25)$. (10 points)

3. Assume that $f \in C^2([a, b])$ and that c is a unique **simple root** of the function $f(x)$, in the interval $[a, b]$, i.e. $f(c) = 0$, $f'(c) \neq 0$. Give Newton's (Newton-Raphson) iterative procedure to find c and prove that, if $\{x_k\}, k = 0, 1, ...\}$ is the sequence of iterants of this method then :

$$\lim_{k \to \infty} \frac{|x_{k+1} - c|}{|x_k - c|^2} = M.$$

where M is a constant depending on the function f. Find M. (10 points)