



**AMERICAN UNIVERSITY OF BEIRUT**  
**Faculty of Arts and Sciences**  
**Mathematics Department**

**MATH 251**  
**FINAL EXAMINATION**  
**FALL 2001**

Closed Book, 2H1/2

**GIVE YOUR ANSWERS ON THE QUESTION SHEET.**  
**SUBMIT YOUR SCRATCH SHEET.**

<b>STUDENT NAME</b>	
<b>ID NUMBER</b>	

In the next 2 problems, we consider the **IEEE single precision floating point number system**,  $\mathbb{F}_s$ , where the storage of a number in  $\mathbb{F}_s$  uses a 32 bits word, in the following way:

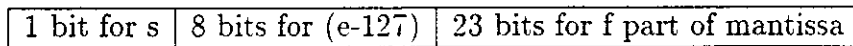


Figure. A word of 32 bits to store  $x = (-)^s(1.f) \times 2^e$ .



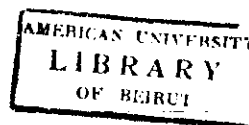
1. Convert the following decimal numbers using the formats indicated in the following table. Fill in the empty cases:

Numbers	Corresponding $\pm(1.f)2^e$ form	Corresponding hexadecimal form
$(-52.234375)_{10}$		
$\frac{1}{4}$		
$\text{succ}(\frac{1}{4})$		
$\text{prev}(\frac{1}{4})$		

**Note:** For  $x \in \mathbb{F}_s$ ,  $\text{succ}(x)$  and  $\text{prev}(x)$  are in  $\mathbb{F}_s$ , the closest numbers to  $x$ , such that  $\text{prev}(x) < x < \text{succ}(x)$ .

2. Fill in the cases to give the decimal values corresponding to the following IEEE single precision representations:

Hexadecimal representation	Fill in decimal values
$[45DE4000]_{16}$	



3. Give (by filling the appropriate case) equivalent mathematical expressions that **avoids loss of significant figures**, when computing the following functions  $f(x)$  for  $x$  taking values in a specific domain.

Case 1		Equivalent form for $f(x)$
$f(x) = \sqrt{1+x^2} - x$	when $x \rightarrow +\infty$	
Case 2		Equivalent form for $f(x)$
$f(x) = \tan(x) - \sin(x)$	when $x$ is near 0	

4. Give the minimum number of arithmetic operations (additions, subtractions and multiplications) to compute the following polynomials

Polynomial $p(x)$	Minimum number of operations
$3x^{32} + x^{65}$	
$6(x-1)^5 + 9(x-1)^9 + 3(x-1)^{17} - (x-1)^{33}$	

5. Consider the function  $f(x) = x^3 - 2\sin(x)$ .

- (a) Graph ON THE SCRATCH SHEET, the function  $f(x)$  on  $(-1.5, 1.5)$ .  
 (b) How many roots does the function  $f(x) = x^3 - 2\sin(x)$  have on  $(-\infty, \infty)$ ?

Number of roots:

- (c) Consider the root  $r$  located on the interval  $[0.5, 1.5]$ .

- Find the least number of iterations that provide an approximation to  $r$  within 7 significant figures using the bisection method.

Number of iterations:

- ⦿ In applying Newton's method to find  $r$ , give the interval  $[a, b] \subset [0.5, 1.5]$ , for which a choice of the initial value  $x_0$  in  $[a, b]$  does not lead to a converging sequence.

Interval  $[a, b]$ :

- Based on the function  $f$ , give the formulae of the iterative schemas that generate the sequence of approximations:  $x_0, x_1, \dots$  for respectively Newton's method and secant method.

Newton's method :	$x_n =$
Secant method	$x_n =$

IN WHAT FOLLOWS CARRY ALL YOUR COMPUTATIONS WITH AT LEAST 5 FIGURES

- Give the sequence of 2 approximations obtained by applying 2 iterations of Newton's method with  $x_0 = 1$ .

$x_1:$	
$x_2:$	

- Give the sequence of 2 approximations obtained by applying 2 iterations of the secant method with  $x_0 = 1.5$  and  $x_1 = 1$ .

$x_2:$	
$x_3:$	

6. A function  $f$  is given on a set of **uniformly spaced data**:  $\{x_i | i = 0, \dots, n\}$ , where  $n$  is a positive integer and  $h = x_{i+1} - x_i$ :

$x_i$	1.8	1.9	2.0	2.1	2.2
$f(x_i)$	3.1268	3.2871	3.4556	3.6328	3.8190

- (a) To interpolate using a degree 3 polynomial for finding an approximation to  $f(1.92)$ , which points would lead to the best approximation?

Points  $x_i$  to be used :

- (b) Obtain an approximation to  $f(1.85)$  using second degree polynomial interpolation. Carry your computations with 5 figures.

Value of approximation to  $f(1.85)$ :

- (c) Assume  $f(x)$  is differentiable up to any order, i.e.  $f \in C^k, \forall k$ . Using Taylor's series expansion around  $x_i$ , give only the **first term** in each of the following:

(i)  $f'_i - \frac{f_{i+1} - f_{i-1}}{2h} = c_1 h^\alpha + c_2 h^\beta + \dots$  and

(ii)  $f''_i - \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} = d_1 h^\gamma + d_2 h^\delta + \dots$

in terms of  $h$  and one of the derivatives of  $f$ .

$f'_i - \frac{f_{i+1} - f_{i-1}}{2h}$	$c_1 =$	$\alpha =$
$f''_i - \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$	$d_1 =$	$\gamma =$

- (d) Using  $\phi_{c,h} = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$ , obtain the best approximation to  $f''(2.0)$  using the above data, successively with  $h_0 = 0.2$  and  $\frac{h_0}{2} = 0.1$ , followed by one Richardson extrapolation. Fill in the empty cases.

$h$	$\phi_{c,h}$	$\phi_{c,h}^{(1)}$
$h_0$		×
		×
$\frac{h_0}{2}$		

- (e) Use Romberg integration with  $h_0 = 0.4$ , to obtain approximations to  $\int_{1.8}^{2.2} f(x)dx$ . Subsequently, fill in the empty cases in the following table:

$h$	$T_h$	$T_h^{(1)}$	$T_h^{(2)}$
$h_0$		×	×
		×	×
$\frac{h_0}{2}$			×
			×
$\frac{h_0}{4}$			

Reminder: The first column is computed using the composite trapezoidal rule; the next two are obtained through Richardson extrapolations.