AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 251
FINAL EXAMINATION
FALL 2001
Closed Book, 2H1/2
GIVE YOUR ANSWERS ON THE QUESTION SHEET.
SUBMIT YOUR SCRATCH SHEET.

STUDENT NAME	
ID NUMBER	

In the next 2 problems, we consider the **IEEE single precision floating point number system**, $\mathbb{F}_s$, where the storage of a number in $\mathbb{F}_s$ uses a 32 bits word, in the following way:

1 bit for s	8 bits for (e-127)	23 bits for f part of mantissa

Figure. A word of 32 bits to store $x = (-)^s(1.f) \times 2^e$.

1. Convert the following decimal numbers using the formats indicated in the following table. Fill in the empty cases:

Numbers	Corresponding $\pm(1.f)2^e$ form	Corresponding hexadecimal form
$(-52.234375)_{10}$		
$\frac{1}{4}$		
$\text{succ}(\frac{1}{4})$		
$\text{prev}(\frac{1}{4})$		

Note: For $x \in \mathbb{F}_s$, $\text{succ}(x)$ and $\text{prev}(x)$ are in $\mathbb{F}_s$, the closest numbers to x, such that $\text{prev}(x) < x < \text{succ}(x)$.

2. Fill in the cases to give the decimal values corresponding to the following IEEE single precision representations:

Hexadecimal representation	Fill in decimal values
$[45DE4000]_{16}$	

3. Give (by filling the appropriate case) equivalent mathematical expressions that **avoids loss of significant figures**, when computing the following functions $f(x)$ for x taking values in a specific domain.

Case 1		Equivalent form for $f(x)$
$f(x) = \sqrt{1+x^2} - x$	when $x \to +\infty$	

Case 2		Equivalent form for $f(x)$
$f(x) = \tan(x) - \sin(x)$	when x is near 0	

4. Give the minimum number of arithmetic operations (additions, subtractions and multiplications) to compute the following polynomials

Polynomial $p(x)$	Minimum number of operations
$3x^{32} + x^{65}$	
$6(x-1)^5 + 9(x-1)^9 + 3(x-1)^{17} - (x-1)^{33}$	

5. Consider the function $f(x) = x^3 - 2\sin(x)$.

(a) Graph ON THE SCRATCH SHEET, the function $f(x)$ on $(-1.5, 1.5)$.

(b) How many roots does the function $f(x) = x^3 - 2\sin(x)$ have on $(-\infty, \infty)$?

Number of roots:	

(c) Consider the **root** r **located on the interval** $[0.5, 1.5]$.

- Find the least number of iterations that provide an approximation to r within 7 significant figures using the bisection method.

Number of iterations:	

- In applying Newton's method to find r, give the interval $[a, b] \subset [0.5, 1.5]$, for which a choice of the initial value x_0 in $[a, b]$ does not lead to a converging sequence.

Interval $[a, b]$:	

- Based on the function f, give the formulae of the iterative schemas that generate the sequence of approximations: $x_0, x_1, \ldots$ for respectively Newton's method and secant method.

Newton's method :	$x_n =$
Secant method	$x_n =$

IN WHAT FOLLOWS CARRY ALL YOUR COMPUTATIONS WITH AT LEAST 5 FIGURES

- Give the sequence of 2 approximations obtained by applying 2 iterations of Newton's method with $x_0 = 1$.

x_1:	
x_2:	

- Give the sequence of 2 approximations obtained by applying 2 iterations of the secant method with $x_0 = 1.5$ and $x_1 = 1$.

x_2:	
x_3:	

6. A function f is given on a set of **uniformly spaced data**: $\{x_i | i = 0, .., n\}$, where n is a positive integer and $h = x_{i+1} - x_i$:

$\mathbf{x_i}$	1.8	1.9	2.0	2.1	2.2
$\mathbf{f(x_i)}$	3.1268	3.2871	3.4556	3.6328	3.8190

(a) To interpolate using a degree 3 polynomial for finding an approximation to $f(1.92)$, which points would lead to the best approximation?

Points x_i to be used :	

(b) Obtain an approximation to $f(1.85)$ using second degree polynomial interpolation. Carry your computations with 5 figures.

Value of approximation to $f(1.85)$:	

(c) Assume $f(x)$ is diffentiable up to any order, i.e. $f \in C^k, \forall k$. Using Taylor's series expansion around x_i, give only the **first term** in each of the following:

(i) $f'_i - \frac{f_{i+1}-f_{i-1}}{2h} = c_1 h^\alpha + c_2 h^\beta + ...$ and
(ii) $f"_i - \frac{f_{i+1}-2f_i+f_{i-1}}{h^2} = d_1 h^\gamma + d_2 h^\delta + ...$
in terms of h and one of the derivatives of f.

$f'_i - \frac{f_{i+1}-f_{i-1}}{2h}$	$c_1 =$	$\alpha =$
$f"_i - \frac{f_{i+1}-2f_i+f_{i-1}}{h^2}$	$d_1 =$	$\gamma =$

(d) Using $\phi_{c,h} = \frac{f_{i+1}-2f_i+f_{i-1}}{h^2}$, obtain the best approximation to $f"(2.0)$ using the above data, successively with $h_0 = 0.2$ and $\frac{h_0}{2} = 0.1$, followed by one Richardson extrapolation. Fill in the empty cases.

h	$\phi_{c,h}$	$\phi^{(1)}_{c,h}$
h_0		$\times$ $\times$
$\frac{h_0}{2}$		

(e) Use Romberg integration with $h_0 = 0.4$, to obtain approximations to $\int_{1.8}^{2.2} f(x)dx$. Subsequently, fill in the empty cases in the following table:

h	T_h	$T_h^{(1)}$	$T_h^{(2)}$
h_0		$\times$ $\times$	$\times$ $\times$
$\frac{h_0}{2}$			$\times$ $\times$
$\frac{h_0}{4}$			

Reminder: The first column is computed using the composite trapezoidal rule; the next two are obtained through Richardson extrapolations.