American University of Beirut
N. Nahlus
Math. 241 Final Examination Feb. 3, 1996
Time: 2.5 hrs.

1. (i) Show that $\mathbf{GL(n, C)/ SL(n, C) \approx C^*}$ by applying the Fundamental Theorem of homomorphisms on the det. map.
(ii) Show that $\mathbf{GL(n, C) = SL(n, C) . C^* I}$ is not a direct product for $n > 1$.

2. Show that $Z_3 \times Z_4 \times Z_5$ is isomorphic to Z_{60}.

3. Let G be a group of order 35.
Let A be a subgroup of G of order 7, and let B be a group of order 5.
(i) show that A is normal in G
(ii) Show that $Z(G) \neq \{e\}$ by applying the *general* class equation on A.
(iii) Conclude that G is abelian as well as cyclic.

4-5. Answer **any 3** parts of the following: (i) Show that U_{17} is a cyclic group.
(ii) Construct a non-abelian group of order 55.
(iii) Classify, up to isomorphism, all abelian groups of order 5000.
(iv) How many conjugates of $a=(1\ 2)o(3\ 4\ 5)$ in S_7, and list 3 of them, **and** how many elements commuting with $a=(1\ 2)o(3\ 4\ 5)$ in S_7, and list 3 of them (Hint: See 6(ii)).

6. Let a be an element of a group G. Let [[a]] be the set of conjugates of a in G, and let C be the centralizer of a in G (i.e, those commuting with a),
(i) Show that the mapping given by $f(gag^{-1}) = gC$ is a well defined bijection from the set [[a]] onto the set of left cosets of C in G,
(ii) Conclude that $o[[a]] = o(G)/o(C)$ if $o(G) < \infty$.

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7. Let G be a p-group, i.e, $o(G) = p^n$ where p is a prime.
If $n \geq 2$, show that G has a normal subgroup of index p^2.
(Hint: Apply induction and the fact that $Z(G) \neq \{e\}$).

8-9. Answer **any 2** of the following:
(i) Let I be an ideal of a ring R. Show that $R/I = \{I+x \ ; \ x \in R\}$ is a ring in a natural way. (Check only that the evident multiplication is well-defined.)
(ii) Construct a field of 49 elements.
(iii) If I is a maximal ideal of a commutative ring R with 1, show that R/I is a field.

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10. Let T be the subgroup of an *abelian* group G of all elements of finite order.
(i) Show that G/T has no elements of finite order (other than the identity element).
(ii) Show that T is fully invariant in G. (Hint: This is very easy and independent of (i)).

11. Show that every finite group other than Z_2 has a non-trivial automorphism.
(Hint: First, assume that G is non-abelian)

12. Let A and B be two normal subgroups of G. In $G/A \times G/B$, consider the subgroup F of all elements (Ax, By) such that ABx = ABy.
Show that $G/(A \cap B) \approx F$.