Math. 241 Final Examination
Time: 2.5 hrs.

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1. Let $f: G \to H$ be a homomorphism of finite groups.
(i) If (o(G), o(H))=1, show that $f(G) = 1_H$
(ii) What can we say in the case where A is a subgroup of G such that (o(A), o(H))=1? Justify your answer.

2. Show that $Z_3 \times Z_4 \times Z_5$ is isomorphic to Z_{60} (as rings).

3. Let G=A.B (direct). Show that $G/A \cong B$

4. Let N be a normal subgroup of G such that (o(N), index(N))=1. If $x \in G$ such that $x^{o(N)} \in N$, show that $x \in N$. (Hint: simplify $(xN)^{o(N)}$ recall index(N)=o(G/N))

5. (i) Construct a non-abelian group of order 21 (by adjoining cyclic groups)
(ii) Classify, up to isomorphism, all abelian groups of order $2^3.5.7^3$.

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6. Let a be an element of a finite group G. Let Conj(a) be the set of conjugates of a in G, and let C be the centralizer of a in G. Show that the mapping given by $f(gC) = gag^{-1}$ is a well defined bijection from the set of left cosets of C in G onto Conj(a). Then deduce |Conj(a)| = o(G)/o(C)

7. Let G be a p-group, i.e, o(G)= p^n where p is a prime.
If n $\geq$ 2, show that G has a normal subgroup of index p.
(Hint: Apply induction and the known fact that Z(G) $\neq$ {e}).

8. Construct a field of 125 elements & justify the theorem you are using.

9. If 0& R are the only left ideals of a ring R with 1, show that R is a division ring

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10. Let R=I+J where I and J are two ideals of an infinite ring R such that I$\cap$J=0 and J is finite. If R=I+J' where J' is an ideal of R such that I$\cap$J'=0, show that o(J)=o(J').

11. Let N be a normal subgroup of G such that (o(N), index(N))=1. Show that N is the only subgroup of order o(N). (Hint: Consider the projection $G \to G/N$)

12. (a) Deduce Cauchy's theorem from the weak version of the 1st Sylow theorem.
(b) If the order of every element of a finite group G is a power of 5, show that o(G) is a power of 5.

13. Show that every non-abelian group of order 6 is isomorphic to S_3.