Math 241 Final Exam. (Prof. N. Nahlus) Jan. *27, 2004*

Part I: (80%) **Answer any 8** of the following 10 problems.

1. Show that $Z_3 \times Z_4 \times Z_{25} \cong Z_{300}$ by finding a (well-defined) isomorphism between the groups.

2. Let H be a normal subgroup of a finite group G where o(G)=170 & o(H) =10 .
If $x^{10} \in H$, show that $x \in H$. (What is o(G/H) ?)

3. Answer any part
(i) Prove the 1st theorem of Sylow (assuming Cauchy's Theorem for abelian groups).
OR (ii) Prove the Orbit-Stabilizer theorem for a finite group G acting on a set S. (|orbit(s)| = ...)

4. Let A and B are normal subgroups of a group G. Show that $AB/A \cong B/(A \cap B)$.

5. Construct a non-abelian group of order 55.
(Hint: Use the format $G = < a, b \mid a^{11}=1, b^5=1, \text{and } bab^{-1} = a^i >$ with a suitable i.)

6. Let G be a group of order 55 such that $Z(G) \neq \{1\}$. Show that G is abelian.
(Hint: G is abelian iff................)

7. Let G be a finite abelian group G of order 100.
Show that the mapping $f: G \to G$ given by $f(x) = x^{17}$ is an isomorphism.

8. Show that every field R has no ideals other than 0 & R.

9. If 0 & R are the only ideals of a commutative ring R with 1, show that R is a field.

10. Prove any part of the following
(i) If I is a maximal ideal of a commutative ring R with 1, show that R/I is a field.
OR (ii) If I is a prime ideal of a commutative ring R with 1, show that R/I is an integral domain.

Part II (20%) **Obligatory**

A) Let H be a subgroup of G of index 7. If 7 is the smallest prime dividing the order of G, show that H is normal in G.
(Hint: Consider the morphism $f: G \to Perm(G/H)$ given by $f(g) = L_g$. Show that $H = Ker f$.)

B) Let A and B be two normal subgroups of G. Consider the subgroup

$$F = \{(xA, yB) : xAB = yAB\}$$

within the group $G/A \times G/B$. Show that $G/(A \cap B) \cong F$.

C) Let $H \leq A \leq G$ where A is a cyclic normal subgroup of a group G.
Show that H is normal in G. (Hint: Every subgroup of a cyclic group is also cyclic)