Time: 2 Hours	MATHEMATICS 241	Name ------------------
January 25, 2006	FINAL EXAMINATION	ID -----------------
Prof. H. Abu-Khuzam.	First Semester 2005-2006	

1. Suppose that a group G is the internal direct product of its normal subgroups H and K. Prove that $G/H \approx K$.

   [9 points]

2. Let G be a finite group of order n. Suppose that $j : G \to H$ is a homomorphism from G onto a group H of order m. Prove that m divides n.

[7 points]

3. Show that the alternating group A_n ($n > 1$) of all even permutations in S_n has order n!/2 by defining a one-to-one and onto mapping from A_n onto the set B_n of odd permutations in S_n.

[7 points]

4. Let G be a group of order pq, where p, q are primes with $p<q$. Prove that G has a <u>normal</u> subgroup of order q

[7 points]

5. Let K be a cyclic normal subgroup of a group G . Prove that every subgroup of K is normal in G.

[7 points]

6. Let G be an abelian group. Let a, b $\in$ G such that o(a)= p, o(b)= q , where p and q are distinct primes. Show that o(ab) = pq.

[7 points]

7. Prove that each homomorphism from a field to a ring is either one-to-one or maps everything onto $\{0\}$.

[7 points]

8. Let R be a finite commutative ring with identity. Prove that every prime ideal in R is a maximal ideal.

[7 points]

9. (a) Let R be ring with identity element 1_R. Let $j : R \to S$ be a ring homomorphism from R onto S. Prove that $j(1_R)$ is the identity of S.

[6 points]

(b) Give an example of a ring homomorphism $j : R' \to S'$ of rings with identity such that $\boldsymbol{j}(1_{R'})\ 1_{S'}$

[6 points]

10. Let R be a commutative ring with identity 1. Let M R be an ideal of R such that R/M is a field. Show that for each $r \in R, r \notin M$, there exists $s \in R$ such that $rs - 1 \in M$

[8 points]

11. Answer **TRUE or FALSE** only: [2 points each, -1/2 point penalty for each wrong answer].

(a) ----- Z_5 is isomorphic to a subgroup of S_5.

(b) ----- Let $j : G \to H$ be a group homomorphism . If G is abelian, then H is abelian.

(c) ----- If G is a finite cyclic group, then G has a prime order.

(d) ----- If $j : Z_7 \to H$ is a nontrivial group homomorphism, then j is one-to-one.

(e) ----- Let S= $\left\{ \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} : a, b, c, d \in Z \right\}$. Then the ideal S is a prime ideal of the ring of 2×2 matrices over Z.

(f) ----- The equation $x^2 - 3x + 2 = 0$ has exactly 2 solutions in the ring Z_6.

(g) ----- The subset S ={(a,a)∈ $Z \times Z$ | a∈Z} of Z×Z is an ideal of Z×Z.

(h) ----- The direct product of two integral domains is an integral domain

(i) ----- No group of order 20 is simple

(j) ----- The permutation (1,4,5,6)(2,1,5) is a cycle in S_6.

(k) ----- If R is a ring of order 7 and I ≠R is a maximal ideal in R, then I ={0}.

[22 points]