Topology I - Math 214 Final

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Exercise 1. Endow $\mathbb{R}^2$ with the Euclidean metric d_2. For every $R > 0$, we denote by $S(0, R)$ the sphere of center the origin and radius R and by $B(0, R)$ the open ball of center the origin and radius R.

1. (a) (General question) Let X and Y be two topological spaces and $f : X \longrightarrow Y$ a map. Show that if X is connected and f is continuous, then $f(X)$ is connected.

   (b) **Using a suitable map** $f : [1, +\infty) \times [0, 2\pi] \longrightarrow \mathbb{R}^2$, show that $B(0,1)^C := \mathbb{R}^2 \setminus B(0, 1)$ is connected.
   One can similarly prove (or deduce) that for every $R > 0$, $B(0, R)^C := \mathbb{R}^2 \setminus B(0, R)$ is connected. You can use this fact later on without producing a proof.

2. (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous map. Let $a \in \mathbb{R}$. Show that if $f^{-1}(\{a\})$ is bounded, then there exists $R > 0$ such that

$$f\left(B(0,R)^C\right) \subseteq (-\infty, a) \quad \text{or} \quad f\left(B(0,R)^C\right) \subseteq (a, +\infty).$$

   (b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous **surjective** map. Show that for every $a \in \mathbb{R}$, $f^{-1}(\{a\})$ is unbounded.
   Hint: For every $R > 0$, $B(0, R)$ has compact closure.

Exercise 2. A topological space X is said to be *compactly generated* if a subset A is closed in X whenever $A \cap K$ is closed in K for every compact subset $K \subseteq X$.

1. (Preliminary) Let X be any topological space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to some $l \in X$. Show that

$$K := \{x_n; n \geq 0\} \cup \{l\}$$

   is compact.

2. Deduce that any topological space that fulfills the conclusion of the sequence lemma (e.g. a metrizable space) is compactly generated.

Exercise 3. We propose to find sufficient conditions for a continuous map between two topological spaces to be closed (recall that a map from a space X to a space Y is said to be closed if the direct image of any closed subset of X is closed in Y).

1. Show that if X is a compact topological space and Y is a Hausdorff topological space, then any continuous map $f : X \longrightarrow Y$ is closed.

2. A map $f : X \longrightarrow Y$ is said to be *proper* if for every compact subset $K \subseteq Y$, $f^{-1}(K)$ is compact.

   (a) (Example) Show that a continuous. map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is proper, if and only if, $\lim_{|x|\to+\infty} |f(x)| = +\infty$.

   (b) Let X be a topological space, Y a Hausdorff space and $f : X \longrightarrow Y$ a continuous proper map.
   Let C be a closed subset of X. Show that for every compact subset $K \subseteq Y$, $f(C) \cap K$ is closed in Y.
   Hint: You can adapt some steps made in Question 1. Better, you can use it directly but in a suitable way.

   Remark and conclusion: Using the terminology of Exercise 2, we deduce that any continuous proper map f from a topological space X to a compactly generated Hausdorff space Y is a closed map.

Exercise 4. Let (K, d) be a compact metric space and $f : K \longrightarrow \mathbb{R}$ a continuous map.

1. Show that f is uniformly continuous on K using two methods:
   (a) Method 1: using a suitable open cover of K.
   (b) Method 2: using the sequential characterization of compactness in metric spaces.
   Hint: Suppose that f is not uniformly continuous. Search then for two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in K with a particular behavior.

2. Application: We take here $K = [0, 1]$. We say that $p : K \longrightarrow \mathbb{R}$ is piecewise-linear if there is a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ of K such that p is linear on the interval $[a_i, a_{i+1}]$ for every $i = 0, \cdots, n$.
   Let $\mathrm{PL}(K) \subset C(K)$ be the set of piecewise linear continuous functions on K.
   Show that $\mathrm{PL}(K)$ is dense in $C(K)$ for the topology of uniform convergence.

Bonus: Using Baire's category theorem in a suitable complete metric space, show that there exists continuous functions on $[0, 1]$ which are nowhere monotone (i.e. not monotone on any subinterval of $[0, 1]$).
Explain why Baire's theorem implies actually that, in some sense to be precised rigorously, "almost all continuous functions on $[0, 1]$ are nowhere monotone".

Bonus bis*: Using Baire's category theorem in a suitable complete metric space, show that there exists continuous functions on $[0, 1]$ which are nowhere differentiable (i.e. not differentiable at any $x \in [0, 1]$).
Explain why Baire's theorem implies actually that, in some sense to be precised rigorously, "almost all continuous functions on $[0, 1]$ are nowhere differentiable".