Mathematics 214, First Semester 1995–96

Final Examination

January 29, 1996, 8:00–10:00

Problem 1. 10 pts.

Prove that if $X \times Y$ is a normal space, then X is a normal space.

Problem 2. 10 pts.

Let X be a Tichonov space. Prove that each subspace of X is a Tichonov space.

Problem 3. 10 pts.

Prove that the countable discrete space $\mathbb{Z}$ can be embedded into each infinite metric space.

Problem 4. 10 pts.

Let X be a normal space and A_1, A_2 and A_3 be closed pairwise disjoint subsets of X. Use the Urysohn Lemma to obtain a continuous function $f : X \to \mathbb{R}$ such that

$f(x) = 2$ for each $x \in A_1$, $f(x) = 5$ for each $x \in A_2$, and $f(x) = -1$ for each $x \in A_3$.

Problem 5. 15 pts.

Let X and Y be topological spaces and $A \subset X$ and $B \subset Y$. Start from the definition of the product topology on $X \times Y$ and give a detailed proof that $\overline{A \times B} = \overline{A} \times \overline{B}$.

Problem 6. 15 pts.

Let X be an infinite set with the co-finite topology (that is a non-empty subset U of X is open iff $X - U$ is a finite set).

(a) Prove that X is separable.

(b) Prove that X is first countable iff X is countable.

(c) Prove that X is compact.

Problem 7. 15 pts.

Write a proof that each subspace of a separable metric space is separable.

Problem 8. 15 pts.

Let $\mathcal{B}$ denote the family of subsets of $[0, 2]$ which consists of:

(i) all usual open subsets of $[0, 2]$, and

(ii) all single point sets $\{\frac{1}{n}\}$ where $n = 1, 2, \dots$

Let $\mathcal{T}$ denote the topology on $[0, 2]$ in which $\mathcal{B}$ is a basis of open sets.

(a) Prove that $([0, 2], \mathcal{T})$ is a regular space.

(b) Prove that $([0, 2], \mathcal{T})$ is a second countable space.

(c) Prove that $([0, 2], \mathcal{T})$ is not a compact space.

Bonus problem. 10 pts.

Find a subset A of the Euclidean plane $\mathbb{R}^2$ which is homeomorphic to the space $([0, 2], \mathcal{T})$ of Problem 8.