MATH 214: FINAL EXAM, JANUARY 28TH 2004

NAME :____.

One piece of advice: half of the exam checks if you are familiar with basic definitions and examples. Assuming you studied for the exam, this should be very easy. I suggest you go through the exam and answer these easy questions first. If you feel something is not stated clearly, come over and ask me. But please - one by one!!!

⋆ Problem #1: __________ (20 points)

⋆ Problem #2: __________ (15 points)

⋆ Problem #3: __________ (25 points)

⋆ Problem #4: __________ (35 points)

⋆ Problem #5: __________ (25 points)

⋆ Problem #6: __________ (30 points)

⋆ Total:____________________________ (150 points)

Problem #1 Topological Spaces / Bases / Subbases.

(1) *(5 points)* Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B} \subseteq \mathcal{T}$. What does it mean when we say $\mathcal{B}$ is a basis for $\mathcal{T}$?

(2) *(5 points)* Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S} \subseteq \mathcal{T}$. What does it mean when we say $\mathcal{S}$ is a subbasis for $\mathcal{T}$?

(3) *(10 points)* Let $\mathcal{S}$ be a collection of subsets of X. Describe the topology $\mathcal{T}_{\mathcal{S}}$ on X for which $\mathcal{S}$ is a subbasis. Then prove $\mathcal{T}_{\mathcal{S}}$ indeed is a topology.

Problem #2 Continuous Maps / Sequences.

(1) *(5 points)* Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a map between topological spaces. Define what it means for f to be continuous.

(2) *(5 points)* Let (a_n) be a sequence of elements of a topological space $(X, \mathcal{T})$ and let $a \in X$. Define what it means $a_n \to a$ in $(X, \mathcal{T})$.

(3) *(5 points)* Let (a_n) be a sequence of elements of $(X, \mathcal{T})$, let $a, b \in X$ and let $a_n \to a$ and $a_n \to b$. Is it possible that $a \neq b$? Justify your claim.

Problem #3 Quotient Spaces / Product Spaces.

(1) *(5 points)* Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a map between topological spaces. Define what it means for f to be a quotient map.

(2) *(5 points)* Give three examples of quotient maps. NO PROOFS please.

(3) *(5 points)* Let $\{(X_\lambda, \mathcal{T}_\lambda) | \lambda \in \Lambda\}$ be a family of (non-empty) topological spaces. Define the Tychonoff topology on the product $\prod_{\lambda \in \Lambda} X_\lambda$.

(4) *(10 points)* Prove the projection maps

$$p_{\lambda_0} : \prod_{\lambda \in \Lambda} X_\lambda \to X_{\lambda_0}$$

are quotient maps. If you find this hard, you may consider a simpler case when $\Lambda = \{1, 2\}$ - for half the credit.

Problem #4 Separation Axioms.

(1) *(5 points)* Define what it means for a topological space $(X, \mathcal{T})$ to be:

i) T_0

ii) T_1

iii) T_2

(2) *(5 points)* Give an example of a space which is NOT T_2. Justify BRIEFLY.

(3) *(5 points).* Define:

i) regular spaces

ii) normal spaces

(4) *(10 points)* Let $(X, \mathcal{T})$ be NORMAL and let $U, V \in \mathcal{T}$ satisfy

$$V \subseteq Cl(V) \subseteq U.$$

Prove there exists $W \in \mathcal{T}$ such that:

$$V \subseteq Cl(V) \subseteq W \subseteq Cl(W) \subseteq U.$$

(5) *(10 points)* Let $(X, \mathcal{T})$ be NORMAL and let $U, V \in \mathcal{T}$ satisfy

$$V \subseteq Cl(V) \subseteq U.$$

Prove there exists $f : X \to \mathbb{R}$ such that

i) $f\big|_V = 1$

ii) $Cl(\{x | f(x) \neq 0\}) \subseteq U$.

Feel free to use part (4) even if you haven't solved it.

Problem #5 Compactness.

(1) *(5 points)* Define what it means for a topological space $(X, \mathcal{T})$ to be:

i) compact

ii) locally compact

iii) Lindelöf.

(2) *(5 points)* Let $(X, \mathcal{T})$ be a topological space. Define its one point compactification $(X^*, \mathcal{T}^*)$.

(3) *(~~10~~ 5 points)* Prove that the topological space $(X^*, \mathcal{T}^*)$ from part (2) is compact.

(4) *(10 points)* Prove that the topological space $(X^*, \mathcal{T}^*)$ from part (2) is Hausdorff *if and only if*

i) $(X, \mathcal{T})$ is Hausdorff and

ii) $(X, \mathcal{T})$ is locally compact.

Problem #6 Connectedness.

(1) *(5 points)* Define what it means for a topological space $(X, \mathcal{T})$ to be:

i) connected

ii) locally connected

iii) path connected

iv) locally path connected

(2) *(10 points)* Define (ONLY DEFINE!) the equivalence relations $\sim_C$ and $\sim_{PC}$ on X such that:

i) the equivalence classes of $\sim_C$ are what we call connected components of X;

ii) the equivalence classes of $\sim_{PC}$ are what we call path connected components of X.

(3) *(5 points)* Prove that the relation $\sim_{PC}$ really is an equivalence relation.

(4) *(5 points)* Give a yes or no answer to this: is it true that connected spaces are always:

i) path connected?

ii) locally connected?

(5) *(5 points)* Give a BRIEF justification of the claim you made in part ii) of the previous question.