January 19, 2005

Mathematics 214, First Semester 2004-2005

Final Examination.

Problem 1. 12 pts. What do you know about compact spaces?

Remind the definition. Give examples of compact and non-compact metrizable spaces. Then give examples of compact and non-compact spaces that are Hausdorff but not metrizable. Finally, quote at least 6 theorems about compact spaces.

Problem 2. 6 pts. (This is a "set theory question".)

Let X and $Y_1, Y_2, \ldots, Y_n$ be sets and let $f : X \to Y_1 \times \ldots \times Y_n$ be a function such that $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ for each $x \in X$. Suppose that $A_i \subset Y_i$ for $i = 1, 2 \ldots, n$. Prove that $f^{-1}(A_1 \times A_2 \times \ldots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \ldots \cap f_n^{-1}(A_n)$.

Problem 3. 6 pts. Let X be a topological space, U be an open subset of X, and D be a dense subset of X. Prove that $\overline{U \cap D} = \overline{U}$.

Problem 4. 6 pts. Let X be a topological space and $A \subset X$.

Let $\{0, 1\}$ be a two-point discrete space. Let $f : X \to \{0, 1\}$ be given by the rules that $f(x) = 1$ when $x \in A$, and $f(x) = 0$ when $x \notin A$.

Prove that f is continuous if and only if A is a closed-open set in X.

Problem 5. 6 pts. Give a detailed outline of a proof that each subspace of a compact metrizable space X must be separable.

Do not provide technical details.

Problem 6. 6 pts. Prove that if $X \times Y$ is a normal space, then X is a normal space.

Bonus. 4 pts. Give an example of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\mathbb{R}^2) = [0, 1)$.

Good Luck!

Time: 150 minutes.