Math 214

Final Exam, 23/01/2008

1. (a) (10 points) Let $X = \{1, 2, 3\}$. Show that the discrete topology is the only topology on X which is T_1.

(b) (10 points) Let X, Y be topological spaces, and let Y be a T_2-space. Assume $f : X \to Y$ is one-one, onto and continuous. Prove that X is T_2, too.

(c) (20 points) Let (X, τ) be a T_2-space and $A \subset X$. Show the following property:

$$x \in \overline{A} \iff x \in A, \text{ or each nhood of } x \text{ contains infinitely many points of } A. \quad (1)$$

Hint : Use the following well-known criterion, (in (2) below, X is a topological space, $A \subset X$, and $x \in X$):

$$x \in \overline{A} \iff U \cap A \neq \emptyset \text{ for every open nhood of } x. \quad (2)$$

2. This exercise illustrates that, *in general*, sequences *alone* are inadequate to characterize topologies. –

Let X be an uncountable set.

(a) (10 points) Show that if X is equipped with the discrete topology, then

(∗) *every convergent sequence is constant from some term on, and such a sequence converges to that constant value.*

(b) (10 points) Show that property **(*)** also holds in the following topology

$$\tau = \{U \subset X : X \setminus U \text{ is at most countable }\}.$$