Math 223 - Advanced Calculus, Spring 2017
Review exercises, part 3

Exercise 1. (1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^k. Suppose that $f(x, 0) = 0$ for all $x \in \mathbb{R}$. Show that there exists a C^{k-1} function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = yg(x, y)$. (*Hint*: consider the function $g(x, y) = \int_0^1 \frac{\partial f}{\partial y}(x, ty)dt$).

(2) Let $\rho(x, y) = y - h(x)$ be of class C^k and put $Z = \{\rho = 0\} \subset \mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^k function vanishing on Z (this means that $f(v) = 0$ for all $v \in Z$). Show that there exists a C^{k-1} function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $f = \rho g$. (*Hint*: similar to (1) consider $g(x, y) = \int_0^1 \frac{\partial f}{\partial y}(x, (1 - t)h(x) + ty)dt$).

(3) Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a C^k function $Z = \{\rho = 0\}$, and suppose that $0 \in Z$ is not a critical point for ρ. Let f be as in (2). Show that in a neighborhood of 0 we have $f = \rho g$ for a certain C^{k-1} function g. (*Hint*: use the implicit function theorem - you can assume that the function obtained by using it is C^k).

(4) With the notation of (3), show that $\nabla f(v)$ is a multiple of $\nabla \rho(v)$ for all $v \in Z$.

Exercise 2. Let $Z = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$. Using the Lagrange multipliers method (established in the sixth assignment, Ex. 4), find the local maxima and local minima on Z of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$:

(1) $f(x, y) = -y$;
(2) $f(x, y) = x + y$;
(3) $f(x, y) = x^2 + y^2 - 2y + 1$.

Exercise 3. Using the Lagrange multipliers method, find the local maxima and local minima on Z of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$:

(1) $f(x, y) = x + y$, $Z = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(2) $f(x, y) = x^2 + y^2$, $Z = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

Exercise 4. Let $U \subset \mathbb{R}^n$ be an open domain, and let $\Omega^p(U)$ be the vector space of p-forms.

(1) Define $\star : \Omega^1(U) \to \Omega^{n-1}(U)$ in the following way: if $\omega = \sum_{j=1}^n f_j dx_j$, $\star\omega = \sum_{j=1}^n (-1)^{j-1} f_j dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n$. Show that $\star$ is a linear isomorphism.

(2) Associate in the usual way a vector field $F = (f_1, \dots, f_n)$ to the 1-form $\omega_F = \sum_j f_j dx_j$. Assume $n = 3$, and let F, G be vector fields. Show that $\star\omega_{F\times G} = \omega_F \omega_G$ (here $F \times G$ is the cross product).

(3) From the definition, the product $\omega(\star\omega)$ is an n-form. Show that $\int_U \omega(\star\omega) \geq 0$ for all $\omega \in \Omega^1(U)$, and that the equality holds if and only if $\omega = 0$.

(4)** Show that $\|\omega\| = \int_U \omega(\star\omega)$ defines a norm on $\Omega^1(U)$.

Exercise 5. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism, and let ω be a p-form on $\mathbb{R}^n$. We say that ω is *ψ-invariant* if $\psi^*\omega = \omega$.

(1) Fix a diffeomorphism ψ. Show that the set of ψ-invariant forms in $\Omega^p(\mathbb{R}^n)$ is a vector space.

(2) Show that if ω, η are ψ-invariant then $\psi\eta$ and $d\omega$ are also ψ-invariant.

(3) Given $\theta \in [0, 2\pi]$, define the (linear) diffeomorphism $\psi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ as $\psi_\theta(x, y) = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y)$ (the *rotation* of angle θ). Show that the 1-form $(x^2 + y^2)(xdy - ydx)$ is ψ_θ-invariant for all $\theta \in [0, 2\pi]$. (*Hint*: the easiest way is to show that $x^2 + y^2$ and $xdy - ydx$ are ψ_θ-invariant, and to use (2)).

(4)* Let ψ be a diffeomorphism satisfying $\psi \circ \psi(v) = v$ for all $v \in \mathbb{R}^n$. Show that any p-form ω can be written as $\omega = \eta_1 + \eta_2$, where $\psi^*\eta_1 = \eta_1$ and $\psi^*\eta_2 = -\eta_2$.

Exercise 6. Let $\phi : [-1, 1] \times [-1, 1] \to \mathbb{R}^3$ be the function defined as $\phi(s, t) = (s + t, s - t, st)$, and put $\Sigma = \phi([-1, 1] \times [-1, 1])$.

(1) Show that Σ is a regular surface and compute the normal vector at any point.

(2) Let ω be the 2-form $\omega = -dx_2 dx_3 - x_1 dx_1 dx_2$. Compute $\int_\Sigma \omega$ (with the orientation on Σ given by ϕ).

(3) Define the vector field F as $F(x_1, x_2, x_3) = (-1, 0, -x_1)$. Check that $\int_\Sigma F = \int_\Sigma \omega$.

Exercise 7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function, and suppose that $f(0) = 0$. Show that there exist continuous functions $g_1, g_2, g_3 : \mathbb{R}^3 \to \mathbb{R}$ such that $f = x_1 g_1 + x_2 g_2 + x_3 g_3$. (*Hint*: fixed $(x_1, x_2, x_3) \in \mathbb{R}^3$ define the function $\ell : \mathbb{R} \to \mathbb{R}$ as $\ell(t) = f(tx_1, tx_2, tx_3)$. Check that $f(x_1, x_2, x_3) = \int_0^1 \ell'(t)dt$. Find $\ell'(t)$ by using the chain rule.)

Exercise 8. Recall that a *regular curve* is a C^1 function $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma'(t) \neq 0$ for $t \in [a, b]$.

(1) Let $h : \mathbb{R} \to \mathbb{R}$ be C^1, and define $Z = \{(x, y) \in \mathbb{R}^2 : y = h(x)\}$. Show that there exists a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(\mathbb{R}) = Z$.

(2) Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be C^1, and let $Z = \{(x, y) \in \mathbb{R}^2 : \rho(x, y) = 0\}$. Assume that $0 \in Z$ and $\nabla\rho(0) \neq 0$. Show that there exists $\epsilon > 0$

and a regular curve $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}^2$ such that $\gamma([-\epsilon, \epsilon]) \subset Z$. (*Hint*: use the implicit function theorem and (1)).

(3)* Let $Z = \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}$. Show that there exists no regular curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(\mathbb{R}) = Z$.

Exercise 9. Let ρ be a C^1 real function in $\mathbb{R}^n$, let $Z = \{\rho = 0\}$ and suppose that $\nabla\rho$ does not vanish on Z. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^1 curve such that $\gamma(t) \in Z$ for all $t \in \mathbb{R}$. Show that $\gamma'(t)$ is orthogonal (w.r.t. the usual Euclidean product) to $\nabla\rho(\gamma(t))$ for all $t \in \mathbb{R}$.

Let us recall some notation: in $\mathbb{R}^n$, we denote by ∇ the operator $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. If f is a C^1 function, the vector field $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is called the *gradient* of f, while if $G = (g_1, \ldots, g_n)$ is a C^1 vector field, the function $\nabla \cdot G = \frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n}$ (also written simply ∇G) is called the *divergence* of G. The composition $\nabla^2 = \nabla \circ \nabla$, also denoted by Δ, is the *Laplace operator* or Laplacian; for any C^2 function f one has $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}$.

Exercise 10. (1) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Suppose that $\int_B g = 0$ for any ball $B \subset \mathbb{R}^n$. Show that $g = 0$.

(2) A C^2 function $h : \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* if $\Delta h = 0$. Let now $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function, and suppose that we have $\int_{\partial B} \nabla f = 0$ for any ball $B \subset \mathbb{R}^n$. Show that f is harmonic.

Exercise 11. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions of class C^2, and let U be a domain of $\mathbb{R}^3$ whose boundary $bU = \Sigma$ is a regular surface. Show that

$$\int_U f\Delta g + \int_U \nabla f \cdot \nabla g = \int_\Sigma f\nabla g.$$

Exercise 12. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ has *compact support* if there exists $R > 0$ such that $f(v) = 0$ for all $v \in \mathbb{R}^n$ such that $\|v\| > R$ (in other words f is zero outside a large enough ball). We also say that a p-form $\omega = \sum_{i_1,\ldots,i_p} f_{i_1\ldots i_p} dx_{i_1} \ldots dx_{i_p}$ has *compact support* if all the functions $f_{i_1\ldots i_p}$ have compact support.

(1) Suppose that $\omega \in \Omega_1(\mathbb{R}^2)$ has compact support. Show that there exists $R > 0$ such that $\int_{B_R(0)} d\omega = 0$.

(2)* Suppose that $f, g : \mathbb{R}^3 \to \mathbb{R}$ have compact support. Show that for $R > 0$ large enough we have

$$\int_{B_R(0)} f\Delta g = \int_{B_R(0)} g\Delta f.$$