**Math 223 - Advanced Calculus, Spring 2017**
**Review exercises, part 2**

**Exercise 1.** For each one of the following quadratic forms, establish whether it is positive or negative (semi)definite, or indefinite:
(1) $Q(x_1, x_2) = -4x_1^2 + 12x_1x_2 - 9x_2^2$
(2) $Q(x_1, x_2) = x_1x_2$
(3) $Q(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$.

**Exercise 2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 - x_1^4 - x_2^4.$$

(1) Show that $v_0 = (0, 0)$ is a critical point of $f$, and $H_f(v_0)$ is positive semidefinite.

(2) Show that $v_0$ is neither a local maximum nor a local minimum of $f$. (*Hint*: consider $f(t, 0)$ and $f(t, t)$ for small $|t|$).

**Exercise 3.** Let $L_1 = \{y = c_1x\}$, $L_2 = \{y = c_2x\}$ be two straight lines of $\mathbb{R}^2$ passing through $v_0 = (0, 0)$, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f(v) = 0$ for all $v \in L_j$, $j = 1, 2$.

(1) Let $w_1 = (1, c_1)$, $w_2 = (1, c_2)$. Show that $J_f(v_0) \cdot w_j = 0$, $j = 1, 2$. (*Hint*: one can use the chain rule for a suitable composition, or the mean value theorem along $L_j$).

(2) Suppose that $c_1 \neq c_2$. Show that $v_0$ is a critical point of $f$.

(3) Does it follow that $v_0$ is extremal for $f$? (*Hint*: consider for instance $f(x, y) = y^2 - (c_1 + c_2)xy + c_1c_2x^2$).

**Exercise 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^\infty$ and suppose that $f(v_0) = 0$. For a given $k \in \mathbb{N}$, define $g = f^{k+1}$. Let $P_k$ be the Taylor polynomial of $g$ about $v_0$. Show that $P_k = 0$.

**Exercise 5.** Compute the second order Taylor polynomial $P_2$ about $v_0 = (0, 0)$ for the following functions:
(1) $f(x_1, x_2) = \log(1 + x_1 + x_2) - \frac{1}{2}(x_1^2 + x_2^2)$
(2) $g(x_1, x_2) = (e^{x_1^2 - x_2^3 + \sin(x_1x_2)} - 1)^2$.

**Exercise 6.** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable.

(1) Suppose that $f$ is a contraction. Show that $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$.

(2) Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial. Show that if $f$ is a contraction then it has degree at most one, and furthermore $|a_1| < 1$.

**Exercise 7.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function of class $C^1$ such that $J_f(v)$ is invertible for all $v \in \mathbb{R}^n$.

(1) Suppose that $n = 1$. Does it follow that $f : \mathbb{R} \to \mathbb{R}$ is invertible?

(2) Same question in the case $n = 2$. (*Hint*: study the function $f(x, y) = (e^x \cos(y), e^x \sin(y))$.)