Math 223 - Advanced Calculus, Spring 2017
Review exercises

Exercise 1. Verify whether or not the following functions are continuous at the point $v = (0, 0)$:

(1) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{x^3y^2}{x^2+y^4}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$;

(2) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{x^2y^2}{x^4+y^4}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$;

(3) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{x^2y^3+2xy^4}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

Exercise 2. Let V be a real vector space, and let $\| \cdot \|$ be a norm on V. Recall that the *ball* of radius $r > 0$ and center $x \in V$ is the set $B_r(x) = \{v \in V : \|v - x\| < r\}$.

(1) Show that $\|av + bw\| \leq |a|\|v\| + |b|\|w\|$ for all $a, b \in \mathbb{R}$ and $v, w \in V$.

(2) A subset U of V is called *convex* if it satisfies the following property: for all $v, w \in U$ and $0 \leq t \leq 1$ we have $tv + (1-t)w \in U$. Show that $B_1(0)$ is convex. (*Hint*: use part (1)).

(3) Show that any ball $B_r(x)$ is convex (*Hint*: one can use for instance (2)).

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x_1, x_2) = \cos(x_1x_2)$. Compute the Hessian matrix of f at a general point $p = (x_1^0, x_2^0) \in \mathbb{R}^2$.

Exercise 4. Let V be a vector space. Recall that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on V are called *equivalent* if there are constants $C, D > 0$ such that $C\|v\|_1 \leq \|v\|_2 \leq D\|v\|_1$ for all $v \in V$.

(1) Let $V = \mathbb{R}^n$ with the standard Euclidean norm $\| \cdot \|$, and standard basis $e_1, \dots, e_n$. Let $\| \cdot \|_1$ be another norm on $\mathbb{R}^n$, and define $M = \max\{\|e_1\|_1, \|e_2\|_1, \dots, \|e_n\|_1\}$. Show that $\|v\|_1 \leq nM\|v\|$ for all $v \in \mathbb{R}^n$. (*Hint*: consider the case $\|v\| = 1$ first).

(2) Let $V = \mathbb{R}^n, \| \cdot \|, \| \cdot \|_1$ be as in (1), and consider in $\mathbb{R}^n$ the Euclidean metric $d(x, y) = \|x - y\|$. Show that the function $\| \cdot \|_1 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz with respect to this metric.

(3) Show that $\| \cdot \|_1$ admits a maximum $D > 0$ and a minimum $C > 0$ on the set $S_1(0) = \{v \in \mathbb{R}^n : \|v\| = 1\}$. (*Hint*: $S_1(0)$ is compact (why?) and $\| \cdot \|_1$ is continuous by (2)).

(4) Using (3), show that $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent. Deduce that if V be a finite dimensional vector space, all norms on V are equivalent.

Exercise 5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous, bijective function with (continuous) inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, we have $f \circ f^{-1}(v) = v$ for all $v \in \mathbb{R}^2$). Suppose that f is differentiable on $\mathbb{R}^2$. Does it follow that f^{-1} is differentiable on $\mathbb{R}^2$? (If yes, give a proof, otherwise find a counterexample).

Exercise 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function, and fix $v \in \mathbb{R}^2$. We define the *directional derivative* $\frac{\partial f}{\partial v}$ as $\frac{\partial f}{\partial v}(p) = \lim_{t\to 0} \frac{f(p+tv)-f(p)}{t}$.

(1) Show that the limit exists, so that $\frac{\partial f}{\partial v}$ is indeed well defined.

(2) Let $J_f(p)$ be the Jacobian of f at p (which is a 1×2 row vector). Show that $\frac{\partial f}{\partial v}(p) = J_f(p) \cdot v$.

(3) Let $p_1, p_2 \in \mathbb{R}^2$ be such that $p_2 - p_1 = v$. Let $S = \{(1-t)p_1 + tp_2 : 0 \leq t \leq 1\}$ (S is the segment with endpoints p_1, p_2). Show that there is a point $q \in S$ such that $\frac{\partial f}{\partial v}(q) = f(p_2) - f(p_1)$. (*Hint*: show that $(1-t)p_1 + tp_2 = p_1 + tv$, define $g : [0,1] \to \mathbb{R}$ as $g(t) = f(p_1 + tv)$ and apply the mean value theorem).

Suggested problems (for those who look for a harder challenge!): Rudin, Chapter 9, ex. 7,8,11,13,14.