Math 223–Final Exam (Fall 01)

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Problem 1
(i) (6 pts) Suppose $F : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions. Define a function $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = F(x,y,h(x,y))$. Find $\partial g/\partial x$ and $\partial g/\partial y$ in terms of $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, $\partial h/\partial x$, and $\partial h/\partial y$.

(ii) (4 pts) Use part (i) to show that the equation $2xy + e^{xz} - z\log y - 1 = 0$ can not be solved for z in terms of (x,y) in a neighborhood of the point $(0,1,1)$.

Problem 2
(12 pts) Let $f(x,y,z) = x\sqrt{y^2+4}$ and consider the 2-surface

$$\phi(s,t) = \left(s, t, 4 - \frac{t^2}{4}\right), \qquad 0 \le s \le 1,\ -4 \le t \le 4.$$

Find $\int_\phi f$.

Problem 3
Let $F(x,y,z) = (y, xz, z^2)$ and consider the 2-surface

$$\phi(s,t) = (s, t, 1-s-t), \qquad (s,t) \in Q^2.$$

(i) (15 pts) Let $\sigma_1 = [0, e_1]$, $\sigma_2 = [e_1, e_2]$, and $\sigma_3 = [e_2, 0]$. Find $\int_{\phi\circ\sigma_1} F$, $\int_{\phi\circ\sigma_2} F$, and $\int_{\phi\circ\sigma_3} F$.

(ii) (15 pts) Find

$$\int_{\partial\phi} F \cdot T\,ds = \int_{\partial\phi} F$$

(a) directly, and (b) using Stokes' formula.

Problem 4 Suppose $R \subset \mathbb{R}^n$ is a rectangle and $f : R \to \mathbb{R}$ is bounded. We define the oscillation of f at a, $\omega(f;a)$, by

$$\omega(f;a) = \lim_{\delta\to 0}\left(\sup_{B(a,\delta)} f - \inf_{B(a,\delta)} f\right).$$

For $\epsilon > 0$, set

$$K_\epsilon = \{a \in R : \omega(f;a) \ge \epsilon\}.$$

(i) (5 pts) Prove that the above limit exists for all $a \in R$ and that $\omega(f;a) = 0$ if a is a point of continuity of f.

(ii) (5 pts) Prove that K_ϵ is compact.

(iii) (8 pts) If the set of discontinuities of f has measure zero, prove that K_ϵ has content zero for every $\epsilon > 0$. (Hint: Start by proving that if a set is compact and has measure zero, then it has content zero.)

Problem 5 For $u \in U = B(0,\sqrt{2}) \subset \mathbb{R}^n$, define $G(u) = (G_1(u), \ldots, G_n(u))$ with

$$G_j(u) = u_j\sqrt{1 - \frac{u_j^2}{4} - \frac{1}{2}(u_{j+1}^2 + \cdots + u_n^2)} \qquad (j = 1, \ldots, n).$$

(i) (6 pts) Prove that G is 1-1.

(ii) (2 pts) Prove that

$$|G_j(u)| = \sqrt{u_j^2 - \frac{1}{4}[(u_j^2 + \cdots + u_n^2)^2 - (u_{j+1}^2 + \cdots + u_n^2)^2]} \qquad (j = 1, \ldots, n)$$

and $\sqrt{1 - |G(u)|^2} = 1 - \frac{|u|^2}{2}$ for all $u \in U$.

(iii) (6 pts) For $x \in X = B(0,1) \subset \mathbb{R}^n$, define $F(x) = (F_1(x), \ldots, F_n(x))$ with

$$F_j(x) = \frac{x_j}{|x_j|}\sqrt{2\left(\sqrt{1 - (x_{j+1}^2 + \cdots + x_n^2)} - \sqrt{1 - (x_j^2 + \cdots + x_n^2)}\right)} \qquad (j = 1, \ldots, n)$$

($F_j(0) = 0$). Prove that $G \circ F(x) = x$ for all $x \in X$.

(iv) (6 pts) Prove that $J_G(u) \neq 0$ for all $u \in U$. Then conclude that $G : U \to X$ is a C^1 diffeomorphism.

(v) (10 pts) Let $G = B(0, \sqrt{3}/2) \subset \mathbb{R}^n$. Use the change of variables $u \to G(u)$ to show that

$$\int_G e^{\sqrt{1-|x|^2}} x_1 \ldots x_n dx = e \int_{G'} e^{-|u|^2/2} \prod_{j=1}^{n} \left(u_i \left(1 - \frac{1}{2}(u_j^2 + \cdots + u_n^2)\right)\right) du$$

for an appropriate set $G' \subset \mathbb{R}^n$.