Math 223–Final Exam (Fall 01) B. Shayya

1. (25 pts) State and prove Fubini's theorem in the case $n = 2$.

2. (25 pts) Suppose $0 < \delta < 1$. Find the surface area of the spherical cap

$$C_\delta = \{\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 : |\theta| = 1 \text{ and } \theta_3 \geq 1 - \delta\}.$$

3. Suppose $h : \mathbb{R} \to \mathbb{R}$ is a positive C^2 function and define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(u, v) = (u, vh(u)).$$

Let

$$\begin{aligned} \Omega_1 &= \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq (1 - x)h(x)\}, \\ \Omega_2 &= \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } (1 - x)h(x) \leq y \leq h(x)\}, \end{aligned}$$

and $\Omega = \Omega_1 \cup \Omega_2$. Also consider the 2-chain

$$\Psi = T\sigma_1 + T\sigma_2$$

in $\mathbb{R}^2$, where $\sigma_1 = [0, e_1, e_2]$ and $\sigma_2 = [e_1 + e_2, e_2, e_1]$ (of course, σ_1 and σ_2 are the oriented affine 2-simplexes in $\mathbb{R}^2$ given by

$$\sigma_1(u, v) = ue_1 + ve_2 = (u, v)$$

and

$$\sigma_2(u, v) = e_1 + e_2 - ue_1 - ve_2 = (1 - u, 1 - v).)$$

(*i*) (5 pts) Prove that the C^2 mapping T is one-to-one and that $J_T(u, v) > 0$ for all $(u, v) \in \mathbb{R}^2$.

(*ii*) (6 pts) Prove that $\text{Im}(T\sigma_1) = \Omega_1$ and $\text{Im}(T\sigma_2) = \Omega_2$.

(*iii*) (9 pts) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, prove that

$$\int_\Psi f(x, y)\, dx \wedge dy = \int_\Omega f(x, y)\, dx\, dy.$$

(*iv*) (10 pts) Find $\partial\Psi$.

(*v*) Find

$$\int_{\partial\Psi} y\, dx + 2x\, dy$$

(a) (12 pts) directly, and (b) (8 pts) using Stokes' theorem.