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AMERICAN UNIVERSITY OF BEIRUT

MATHEMATICS 223, FINAL EXAMINATION

FALL SEMESTER, 2000-1

Answer the following questions:

1. (a) Define uniform continuiuty for functions between metric spaces.

(4 pts.)

(b) Suppose f is a uniformly continuous function on the segment $(0,1)$.

(i) Prove that if $\{x_n\}$ is a Cauchy sequence in $(0,1)$, then $\{f(x_n)\}$ is also a Cauchy sequence. (6 pts.)

(ii) Use (i) to show that f extends continuously to the closed interval $[0,1]$; that is, there exists a continuous function g of $[0,1]$ such that $f(x) = g(x)$ for every $x \in (0,1)$. (6 pts.)

2. (a) Define discontinuities of the first and second kind. (4 pts.)

(b) Prove that a monotone function f on a segment (a,b) has no discontinuities of the second kind.

(6 pts.)

(c) Prove that the derivative of a differentiable function f on a segment (a,b) has no discontinuities of the first kind.

(6 pts.)

3. (a) Let f be a real function that is continuous on $[a, b]$ and differentiable in (a, b). Show that there exists $c \in (a, b)$ such that $f(b) - f(a) = cf'(c)\ln(b/a)$. (Hint: Use the generalized mean-value theorem) (6 pts.)

(b) Show that $\lim_{k\to\infty} k(a^{1/k} - 1) = \ln a$. (4 pts.)

4. (a) Use the partial summation formula to prove that if Σa_n converges, and if $\{b_n\}$ is a monotone bounded sequence, then $\Sigma a_n b_n$ converges. (6 pts.)

(b) Use Schwarz's inequality to prove that if Σa_n, $a_n \geq 0$, converges then $\Sigma\sqrt{a_n}/n$ converges. (4 pts.)

(c) Find the radius of convergence of the series $\Sigma 2^n z^n/n!$. (4 pts.)

5. (a) Define the Riemann-Stieltjes integrability. (4 pts.)

(b) Suppose $f(x) = 0$ for all rational x, and $f(x) = 1$ for all irrational x. Prove that f is not Riemann integrable on $[0, 1]$. (6 pts.)

6. (a) State a criterion for a function f defined on $[a, b]$ to be Riemann-Stieltjes integrable on $[a, b]$.

(4 pts.)

(b) Suppose f is a bounded function on $[-1, 1]$, $\alpha(x) = 0$ if $x < 0$, $\alpha(x) = 1$ if $x > 0$, and $\alpha(0) = 1/2$. Prove directly that $f \in \mathcal{R}(\alpha)$ on $[-1, 1]$ if and only if f is continuous at 0. (12 pts.)

7. (a) State Taylor's theorem. (4 pts.)

(b) Suppose f is a twice-differentiable function on the segment $(0, \infty)$.

(i) Use Taylor's theorem to prove that for $x > c$ and $h > 0$

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. (4 pts.)

(ii) Let M_0, M_1 and M_2 be the least upper bounds of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$, respectively, on (c, ∞). Use (i) to prove that $|f'(x)| \leq hM_2 + M_0/h$.

(4 pts.)

(iii) Use (ii) with $h = \sqrt{M_0/M_2}$ to prove that $M_1^2 \leq 4M_0M_2$. (4 pts.)

(iv) Use (iii) to show that if $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f'(x) = 0$.

(4 pts.)

8. (a) Let X be an infinite set. For $p, q \in X$, define $d(p, q) = 1$ if $p \neq q$, and $d(p, q) = 0$ otherwise. Prove that d is a metric. Which subsets of the resulting metric space are open? which are closed? which are compact?

(6 pts.)

(b) Prove directly from the definition of compactness that the interval $(0, 1]$ is not compact. (6 pts.)

(c) Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 2 and 5. Is E dense in $[0, 1]$? Is E compact? Is E connected? Is E perfect? (6 pts.)