AMERICAN UNIVERSITY OF BEIRUT

MATHEMATICS 223, FINAL EXAMINATION

FALL SEMESTER, 2003-04

Answer the following questions:

Grade Distribution: $x(a) = 2$ **points**, $x \neq 9, 12$; $x(a) = 4$ **points**, $x = 9, 12$; $x(b) = 6$ **points**, $x \neq 9, 12$; $x(b) = 4$ **points**, $x = 9, 12$.

1. (a) Define a metric space.

(b) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define

$$d(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x} - \mathbf{y}|}{1 + |\mathbf{x} - \mathbf{y}|}.$$

Show that $(\mathbb{R}^n, d)$ is a metric space.

2. (a) Define a compact metric space.

(b) Show that the standard Cantor's set is perfect.

3. (a) Define a connected subset of a metric space.

(b) Show that the square $[0, 1] \times [0, 1]$ is a connected subset of $\mathbb{R}^2$.

4. (a) Define a convex subset of $\mathbb{R}^n$.

(b) Show that the closure of a convex subset of $\mathbb{R}^n$ is convex.

5. (a) Define a continuous function between metric spaces.

(b) Show that if f and g are continuous functions from a metric space X to a metric space Y, then the set $E = \{x \in E : f(x) = g(x)\}$ is closed.

6. (a) Define a uniformly continuous function between metric spaces.

(b) Show that a uniformly continuous real function f on a segment $]a, b[$ is bounded. Would this be true if $]a, b[$ is replaced by $\mathbb{R}^1$?

7. (a) Define the norm $\|\ \|$ of a linear transformation $A \in L(\mathbb{R}^n, \mathbb{R}^m)$.

(b) Show that every linear transformation $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ is uniformly continuous.

8. Let $E \subset \mathbb{R}^n$ be open, $\mathbf{x} \in E$, and $f : E \to \mathbb{R}^m$.

(a) Define $f'(\mathbf{x})$.

(b) Show that if $f'(\mathbf{x})$ exists, then $f'(\mathbf{x})$ is unique.

9. (a) Prove that for every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ there exists a unique $\mathbf{y} \in \mathbb{R}^n$ such that $Ax = x \cdot y$. Show that $\|A\| = |y|$.

(b) Prove that if f is a differentiable real function of an open set $E \subset \mathbb{R}^n$ with a real extreme value at a point $\mathbf{x} \in E$, then $f'(\mathbf{x}) = 0$.

10. (a) Define the integral $\int_{I^n} f \, d\mathbf{x}$, where f is a real continuous function on the $n-$cell I^n.

(b) State and prove a sufficient condition for a primitive map G of an open set $E \subset \mathbb{R}^n$ to be locally one-to-one at a point $\mathbf{x} \in E$.

11. (a) State the Inverse-Function theorem.

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^{x^2 - y^2} \cos 2xy, e^{x^2 - y^2} \sin 2xy)$. Show that f is locally one-to-one at every point (x, y) other than the origin. Is f one-to-one in $\mathbb{R}^2$?

12. Define $f : \mathbb{R}^3 \to \mathbb{R}^1$ by $f(x, y, z) = x^2y + e^x + z$.

(a) Show that there exists a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and $f(g(y, z), y, z) = 0$.

(b) Find $\partial g/\partial y(1, -1)$ and $\partial g/\partial z(1, -1)$.

Good Luck