Problem 1 Let K be the convex set spanned by the points

$$p_0 = (0,0,1), \qquad p_1 = (6,0,1), \qquad p_2 = (0,6,1), \quad \text{and} \quad p_3 = (6,0,2)$$

in $\mathbb{R}^3$. Also, let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $F(x,y,z) = (x+y, x+y+z, z-1)$.

(i) (4 pts) Show that K has a positively oriented boundary.

(ii) (18 pts) Find

$$\int_{\partial K} F$$

(a) directly, and (b) using the divergence theorem.

Problem 2 (18 pts) Show that the system of equations

$$\begin{aligned} y_1 \sin y_2 + e^{y_2} + 7x_1^2 - x_2 + x_2 x_3 &= 38 \\ e^{y_1 - 1} + y_2 + x_1 y_2 + 2x_2 - x_3 &= -3 \end{aligned}$$

can be solved for (y_1, y_2) in terms of (x_1, x_2, x_3) in a neighborhood of the point $(2, -3, -2, 1, 0)$. Find an explicit approximate solution.

Problem 3
(i) (7 pts) Suppose $K \subset \mathbb{R}^n$ is compact and has measure zero. Show that K has content zero.

(ii) (5 pts) Does the result of part (i) remain true if we remove the assumption that K is compact?

Problem 4
(i) (7 pts) Suppose $\Omega \subset \mathbb{R}^n$ is open, $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function, and $n > 1$. Show that f is not one to one.

(ii) (5 pts) Suppose that $\Omega \subset \mathbb{R}^n$ is open, $f : \mathbb{R}^n \to \mathbb{R}^k$ is a C^∞ function, and $n > k$. Show that f is not one to one.

Problem 5 (16 pts) Suppose $\Omega \subset \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}$ is C^1, $p \in \Omega$, and $\nabla f(p) \neq 0$. Prove that there are neighborhoods U and V of 0 and p respectively and a C^1 diffeomorphism $G : U \to V$ with $G(0) = p$ and

$$f \circ G(x) = f(p) + x_n$$

for all $x = (x_1, x_2, \ldots, x_n) \in U$.

Problem 6 Let

$$U = \{(u, v, r, s) \in \mathbb{R}^4 : 0 < v < u < \frac{\pi}{2} \text{ and } 0 < s < r\}.$$

For $(u, v, r, s) \in U$, define

$$G(u, v, r, s) = \left(r \cos u + s \cos v,\ r \sin u + s \sin v,\ r^2 + s^2,\ \arctan \frac{r}{s} \right).$$

(i) (4 pts) Prove that G is 1-1.

(ii) (4 pts) Find J_G.

(iii) (4 pts) Let $X = G(U)$. Show that X is an open subset of $\mathbb{R}^4$ and that $G : U \to X$ is a C^1 diffeomorphism.

(v) (8 pts) Suppose $\Delta \subset U$ is compact, the topological boundaries of Δ and $G(\Delta)$ have content zero, $f, F : \mathbb{R}^4 \to \mathbb{R}$ are continuous functions, and

$$\int_{G(\Delta)} F(x, y, z, w)\, dxdydzdw = \int_{\Delta} f \circ G(u, v, r, s)\, dudvdrds.$$

Find $F(x, y, z, w)$ in terms of $f(x, y, z, w)$.
Hint. If $(x, y, z, w) = G(u, v, r, s)$, then

$$x^2 + y^2 = r^2 + s^2 + 2rs \cos(u - v), \qquad r = \sqrt{z} \sin w, \qquad \text{and} \qquad s = \sqrt{z} \cos w.$$

Also, if $0 \leq \theta \leq \pi/2$, then

$$\sin \theta = \sqrt{(1 - \cos \theta)(1 + \cos \theta)}.$$