American University of Beirut *Feb. 4, 1997*
Math 261 Final Examination (Time: 2.5 hours)
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1. If m is square free and $kj \equiv 1 \bmod \phi(m)$, show that $a^{kj} \equiv a \bmod m$ for all integers a, and why this is important for *coding theory* ?

2. Find all primitive solutions in positive integers of the equation
   $x^2 + 7y^2 = z^2$ if y is even.

3. If p is an odd prime and (abc, p) = 1, show that the number of solutions (x, y) modulo p of the equation $ax^2 + by^2 \equiv c \bmod p$ is $p - (\frac{-ab}{p})$.

Hint: $\sum_{y=1}^{p} (\frac{y^2 + k}{p}) = -1$ if $(k, p) = 1$.

4. Find the number of integers n, $1 \leq n \leq p$ such that $(\frac{n}{p}) = (\frac{n+2}{p}) = -1$

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5. If p is a prime, $p \equiv 1 \bmod 4$, and $p = a^2 + b^2$ with a odd and positive, show that $(\frac{a}{p}) = 1$. (Hint: Use the reciprocity law of the Jacobi symbol)

6. Use the Chinese Remainder theorem to show that $x^2 \equiv a \bmod p_1 p_2 \ldots p_k$ is solvable if $x^2 \equiv a \bmod p_i$ is solvable for each i. ($p_1, p_2, \ldots, p_k$ are distinct primes)

7. Compute $(\frac{5}{13})$ by Gauss Lemmas 1 and 2.

8. Find all integer solutions of 10x+ 14y + 35z = 1.

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9. Show that there are infinitely many primes of the form 8k+1 .
   Hint: Every prime divisor of $x^4 + 1$ is of the form 8k+1. (**prove it** !)

10. Let p and q be odd primes such that q = 2p+1 and $p \equiv 1 \bmod 4$. Show that 2 is a primitive root modulo q.

Hint: Find $(\frac{2}{q})$ from p=4k+1. Then apply Euler's Criterion.

11. Let p be an odd prime and (a, p) = 1. Show that $x^n \equiv a \bmod p^m$ is solvable if
   $a^{\frac{\phi(p^m)}{d}} \equiv 1 \bmod p^m$ where $d = g.c.d(n, \phi(p^m))$

Hint: Let g be a primitive root modulo p^m, $x = g^t$ and $a = g^b$. Then solve for t.

12. Show that the number e is irrational.

Hint: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!} + \ldots$ If $e = \frac{i}{k}$, multiply both sides (of the above equation) by k! . Then discover the rest of the proof.

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13. Prove the reciprocity law of the Legendre symbol using *Gauss sums* knowing the fact concerning G^2.