Mathematics 261 Final Examination (Time: 120 minutes) Feb 1, 2006

--Prof. Nahlus--

Part 1 (80%) Answer any 8 problems out of 10 problems

1. Show that $Z[\sqrt{-2}]$ is a Euclidean domain

2. Show that $Z[\sqrt{-5}]$ is not a UFD. (Hint: $(1+\sqrt{-5})(1-\sqrt{-5}) = 6$)

3. Use the Mobius Inversion Formula to solve for ***f(n)*** in each of the following cases.

$(i)\ n^3 = \sum_{d|n} f(d)$ $(ii)\ m(n)n^3 = \sum_{d|n} d^3 f(d)$ (Hint: Make a substitution)

4. If p is a prime, a is positive, a ≡ 1 mod 8, and $p = a^5 + 20ab - 2b^2$.

Show that $(\frac{a}{p}) = 1$. (Hint: Use the reciprocity law of the Jacobi symbol)

5. Find all primitive solutions in positive integers of the equation

$x^2 + 11y^2 = z^2$ if y is even (x & z are odd) . (Hint: $11y^2 = ...$)

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6. State & prove the theorem about the solvability of $x^n \equiv a \mod p$ (where p is an odd prime)

7. Find $|\, x^4 \equiv -6 \mod 75 \,|$ (the number of solutions mod 75).s

8. Suppose $f(a) \equiv 0 \mod p^{25}$ (where p is an odd prime). If $p^5 \parallel f'(a)$,
show that a can be lifted to a solution of $f(x) \equiv 0 \mod p^{40}$
(Hint: $f(a + tp^{20})$)

9. Let p be an odd prime

(i) $p \mid x^4 + 1 \Rightarrow p \equiv 1 \mod 8$

(ii) Deduce that there are infinitely many primes $p \equiv 1 \mod 8$

10. Let p be a prime such that $p \equiv 1 \mod 4$. Show that $p = a^2 + b^2$ for some integers a, b.
(Hint: Consider $x^2 \equiv -1 \mod p$ & Z[i])

Part 2: Obligatory (20%)

A) If m is square free & $k \equiv 1 \mod f(m)$, show that $a^k \equiv a \mod m$ for all a.
(Hint: Write m as a product of distinct primes)

B) Show that $\frac{x^2 - 2}{2y^2 + 3}$ is never an integer for all integers x & y

C) If $q = 4^n + 1$ (n is a positive integer) and $3^{(q-1)/2} \equiv -1$ **(mod *q*),** show that q is a prime.